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A. G. Bagdoev

NONLINEAR PROBLEMS OF COMPRESSIBLE FLOW

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A. G. Bagdoev

NONLINEAR PROBLEMS OF COMPRESSIBLE FLOW

(Nekotorye nelineinye zadachi o dvizhenii szhimaemoi zhidkosti)

Izdatel'stvo Akademii Nauk Armyanskoi SSR
Erevan 1967

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TABLE OF CONTENTS

INTRODUCTION 1
Formulation of the problem 5
Chapter I. FLUID PRESSURE FOR WAVE FRONT MOVING WITH CONSTANT VELOCITY OVER THE SURFACE. 10
§ 1. Linear treatment of the two-dimensional problem. 10
§ 2. Solution of the nonlinear problem by means of simple waves. . 14
§ 3. Pressure near the front of a sound wave 21
§ 4. Solution near the point of contact of the regions and in the whole region 33
§ 5. Solution of the axisymmetric problem 54
§ 6. Validity of the Lighthill—Whitham method for the second approximation 61
Chapter II. PRESSURE DISTRIBUTION FOR ARBITRARY CONDITIONS AT THE BOUNDARY OF A HALF-SPACE 65
§ 1. Formulation of the problem. 65
§ 2. Solution in the linear case and considerations regarding the solution of the nonlinear problem on the shock wave. 66
§ 3. Neighborhood of the sonic line 73
§ 4. Shock wave AB in the nonlinear case 78
§ 5. Pressure on AB by the method of characteristics 83
§ 6. Motion of the shock wave for large times. 87
§ 7. Transition of the pressure front surface velocity through the velocity of sound. 94
§ 8. Pressure at a finite point (x, y) for large times 102
§ 9. Pressure in the linear and nonlinear formulations for an inhomogeneous compressible fluid 103
APPENDIX I 121
APPENDIX II 124
APPENDIX III 133
APPENDIX IV. Linear and nonlinear problems of the motion of a conducting and viscous fluid in a half-space 135

BIBLIOGRAPHY 165

EXPLANATORY LIST OF ABBREVIATED NAMES OF USSR INSTITUTIONS AND PERIODICALS APPEARING IN THE TEXT

Abbreviation	Full name (transliterated)	Translation
DAN SSSR	Doklady Akademii nauk SSSR	Proceedings of the Academy of Sciences of the USSR [journal]
LGU	Leningradskii gosudarstvennyi universitet	Leningrad State University
MGU	Moskovskii gosudarstvennyi universitet	Moscow State University
PMM	Prikladnaya Matematika i Mekhanika	Applied Mathematics and Mechanics [journal]
PMTF	Prikladnaya Mekhanika i Tekhnicheskaya Fizika	Applied Mechanics and Technical Physics [journal]
ZhETF	Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki	Journal of Experimental and Theoretical Physics

INTRODUCTION

The motion of a half-space of ideal compressible fluid due to an arbitrary pressure front moving over its surface is considered in this book. Physically, such a situation arises in an explosion on or above the surface of a fluid. Since a liquid is a considerably denser medium than air, the pressure in the air may be considered as in the reflection of a shock wave from a rigid wall. Consequently, the pressure at the liquid surface will be known as a function of the time and coordinates. The problem consists in finding a solution of the gas dynamic equations for given discontinuity conditions at the boundary.

For a liquid under pressures of the order of several hundreds of atmospheres, the parameter $\frac{P}{\rho_0 a_0^2}$ determining the liquid motion will be small, and therefore a linear theory can be used. The linear theory, however, gives an incorrect picture in the vicinity of the sound wave, as well as near the free surface and at large times. In addition, it is of both theoretical and practical interest to calculate approximations higher than the first.

In the present work the problem is formulated nonlinearly for the pressure, a small parameter $\gamma = \frac{P}{\rho_0 a_0^2}$ is introduced and an isentropic approximation used. It is shown that the linear theory can be greatly improved by using only the linear solution. An investigation of the linear two-dimensional problem by numerical methods when the motion has similarity in time, as well as of a number of nonlinear problems is given in /17/.

In /27/ the solution of the linear problem is considered for an ideal homogeneous compressible fluid with an arbitrary boundary pressure; general formulas are obtained for the solution in a half-space, simplifications are made near the wave fronts and results of numerical computations are given for the pressure distribution in a half-space. The publication also contains a solution of the linear problem for an inhomogeneous, two-component fluid and for an elastic, solid medium; the nonlinear problem of a strong explosion is considered there, and the solution determined in terms of simple waves for a sharply varying pressure behind the front.

In this work various methods of improving the linear solution are studied, and the pressure is determined in the second approximation.

The proposed improvement of the linear solution is based on a Lighthill—Whitham technique for rendering approximate solutions to physical problems uniformly valid, and the short-wave approximation of Khristianovich et al. Both methods are used jointly, enabling a valid solution to be obtained in the required approximation. It is noteworthy that the combination of these

methods results in the well-known ideas /45/ about using a slowly varying amplitude. In Chapter I, with the exception of § 4, the situation is such that the velocity of the pressure front over the surface and the magnitude of the pressure on the front are constant, corresponding to an explosion at the surface. In addition to its independent interest, the mathematical formulation of this problem is similar to that for conical flows past wings (Appendix II) and the diffraction by a wedge and a cone. In a spherical explosion the problem has axial symmetry, whereas in a cylindrical explosion it is two dimensional.

In §§ 1, 2, 3, 4 of Chapter I the main topic is the plane problem. The linear solution in the disturbed region $AB_1B_1'A'$ (Figure 5 on p. 23) is considered, and in § 2 the variables $\xi = \frac{x}{t}$, $\eta = \frac{y}{t}$ are used to deal with the nonlinear isentropic problem in region ABC, where equations (F) (p. 9) are hyperbolic. In the linear case of § 1 the pressure in region ABC is constant along the characteristics, and in the nonlinear problem of § 2 the solution is also looked for in terms of simple waves with constant parameters along the characteristics. In this latter case the conditions (G) (p. 9) at the shock front are satisfied to second order, and one can construct the solution up to the envelope of the characteristics, on which the acceleration is infinite. In the first approximation on the shock wave AB the solution is given by the linear treatment, but for pressures falling sharply behind the front at the fluid boundary an effect different from the linear is obtained /27/. In § 3 the sonic line BB' is considered, on which the pressure vanishes in the linear case /27/ due to a branch in its vicinity. Lighthill's technique /3, 26/ is used to eliminate this singularity in the linear solution; it consists of expressing the required functions in addition to the independent variable as series in a small parameter. This method is fairly general for linear problems with a branch; however, in /21/, its rigor to second order was doubted, and it is also unclear how to apply it to the axisymmetric problem. In the present work this technique is therefore combined with a method used by Whitham mainly in first approximation problems /4/; this consists in replacing the linear characteristic variable $t - \frac{r_1}{a_0}$ in a linear solution by y_1, where $y_1 = \text{const}$ is the equation of the nonlinear characteristics. In § 3 of Chapter I this method yields a solution near the wave front BB' which agrees with Lighthill's solution. In § 5 the corresponding solution is obtained for the axisymmetric problem. In § 6 simplifications corresponding to the short-wave approximation are applied to the equations of gas dynamics near the front BB', and with the aid of the obtained solutions it is shown that the solutions of §§ 5 and 3 satisfy the equations of gas dynamics. The solution in the form (6.5) was first found by Sedov /1/. In accordance with the ideas of using a slowly varying amplitude, the constants in the solutions (6.2) and (6.5) can be determined at the exit from the wave region, that is, for large δ, where the solution should revert to the linear. Thus, a uniformly valid solution is of varying order when moving from the front to the interior of the region. The same refers to the wave front BB', where the solution in the variables ξ, η varies from second to first order of smallness when approaching the point B, which connects the elliptic and hyperbolic regions on the shock front.

In § 3 the procedure of /4/ is applied to the whole vicinity of the sonic line. All the results of /3/ on conical flows are derived, including the strength of a shock wave which can arise ahead of the parabolic line BC, in terms of the variables ξ, η (Figure 1). Near the parabolic line the solution is improved to second order (expressions (3.36), (3.37) in Chapter I). It should be noted that in the problems of Chapter I a shock wave does not arise ahead of BC, since behind BC there is a rarefaction region, according to (3.28). In Appendix II we consider an example of conical flow past a wing of arbitrary cross section perpendicular to the flow, in which a shock wave arises ahead of BC. At a distance $\theta - \theta_0 \sim 1/\sqrt{\gamma}$ in front of the point B, the flow will be two-dimensional in r_1 and θ. The works /8, 21/, which discuss this problem, conclude incorrectly that the pressure jumps discontinuously at the point B when moving along the shock AB toward the $0y$ axis. This problem is analyzed in § 4 of Chapter I and Appendix II. It is shown that the solution in this region will be a short wave dependent on two variables; in the two-dimensional problem the pressure is of order 1, and in the axisymmetric problem its order is $\frac{4}{3}$ /37/. To determine the solution in the two-dimensional problem, a particular exact solution of /9/ is used. However, in contrast to the processes used in /9/ to search for the constants, we chose them by imposing the condition that the solution becomes linear at the exit from the wave region. In this way one may construct a uniformly valid solution in the whole region near the wave and on the shock wave itself. The theory of singular integrals is applied to determine the pressure in the whole region ABC in the second approximation. Without modifying the coordinates by Lighthill's technique /41/ it is difficult to determine the approximations, since the conditions on the shock wave for the potential may not be satisfied. In § 4 of Chapter I the potential and the independent variable t (i.e., the time) are expressed as a series and a solution is found in the whole region ABC in the second approximation. The condition that the potential vanishes on the shock front is satisfied if the second-order term in γ in the expression for t is taken as the first-order improvement to the linear equation of the shock wave. Near the shock front the quadratures for the second approximation are simplified, and the pressure on AB obtained as a single quadrature involving the pressure f_1 and its derivative f_2 at the front in the linear problem (formula (4.57)). The resulting solution is valid for the problems of Chapter I, and also for those of Chapter II where use is made of the front equation in the linear problem to derive the corresponding formulas. The material is divided into two chapters because in this way the problems can be discussed in the second approximation at a point without separating cases of constant and varying boundary conditions.

The vicinity of the shock front AB is calculated in the axisymmetric (§ 5, Chapter I) and general (§ 5, Chapter II) problems by substituting the linear solution in the equations of the characteristics. The form of the shock wave is determined to first order, and then a solution, exact to second order, is found with the aid of the characteristics.

The pressure inside the elliptic region is considered in the second approximation. In addition to deriving the pressure found by Sagomonyan /17/ in a linear approximation, the velocity components with which to find

the second approximation are also determined. Further, for a subsonic velocity at the surface one can apply techniques used in the solution of flow problems around a triangular wing with a subsonic edge /28, 29/.

The second chapter contains a solution of the general problem of the motion of a half-space under a given pressure. An important fact is that for an explosion at the boundary the initial velocity of the pressure front over the surface is infinite, and the shock AB (Figure 1) in the linear problem reaches the axis of symmetry. The segment BB', which corresponds to the region of the initial values on the shock wave $ABB'A'$, and its associated difficulties are thus missing. At the point B on the shock wave the pressure in the linear problem is continuous, in contrast to the situation in Chapter I. Near BC (Figure 1) the branch in the expansion term of the pressure is also removed (§ 3, Chapter II), and, according to the results in § 3 of Chapter I, a shock wave does not form near BC to second order. Another important point in the general problem is the transition of the front velocity over the surface through the sonic value /33/. The resulting irregular reflection near the surface is investigated qualitatively (§ 7, Chapter II) with the aid of simple solutions /6, 10/.

The shock wave attenuation is considered for large times. The linear solution for large times near the wave front is improved by replacing the linear characteristics by more exact ones; the relations on the shock front then yield a law governing the attenuation, the form of which was known /14, 13, 5/, but whose coefficients are now completely specified (§ 6, Chapter II). The method of /5/ is followed, as it is more suitable for our problem. In /5/, and all the more so in Volume 4 of /11/, there is no mention of simply replacing the characteristic variable in the linear solution to obtain an explicit nonlinear solution; instead, a cumbersome substantiation is given of deriving the general solution from the exact equations of the characteristics without any indication how to approach the initial source. It is true that in a later work /43/, which came to our notice after constructing the solution in § 6 of Chapter II, the indicated result is given and the solution verified in another way. We obtain a solution by replacing the linear characteristic variable $t - \frac{r_1}{a_0}$ by y_1, where $y_1 = \text{const}$ is the equation of the improved characteristics (3.2). (The same solution can be found using an integrating factor /14/.) It is then shown that this solution satisfies the equations of gas dynamics, which in the given case are the equations of one-dimensional short waves. The form of the wave profile in our case (Figure 14'; Appendix I) differs from that of the expansion of a spherical piston /5/. In Appendix I the rear discontinuity is determined, its position being found from the condition that the areas between the curve $F(y_1)$ and the secant are equal. The motion with time on the shock wave AB, with the exception of the vicinity of the free surface, will be given by one-dimensional motion along rays. The improved solution is found for an inhomogeneous compressible fluid near a corner of the front (Figure 17). The pressure distribution near the wave front BB' for an inhomogeneous fluid is determined by previous methods; in the linear solution (9.7) of Chapter II near the wave front the linear characteristic $\tau = t - \int \frac{ds}{a(y)}$ is replaced by y_1, where $y_1 = \text{const}$ is the equation of the more exact characteristics. The equations of gas dynamics are then written in the variables

τ, s where the dependence on s is slow compared with τ. For simplicity the solution is considered on the $0y$ axis, but is valid along the whole of BB' if y is replaced by s and $\int_0^y a(y)\,dy$ by H_2. The solution derived from the simplified equations depends on an arbitrary function, whose form is determined at the transition to the linear solution. Thereafter, the solution agrees with that obtained by Whitham's method. The technique of replacing the linear characteristics by the more exact ones thus gives the solution near the front BB' in the second approximation.

The use of Lighthill's technique /3/ in the given case results in another solution. The method of /18/ cannot be directly applied to the vicinity of the wave BB', since it is unclear how the method of characteristics can be applied to derive the linear solution (9.7) in Chapter II.

In this connection it is of interest to mention the attempt to obtain linear solutions near BB' not from the integral formulas /27/, but by asymptotic methods used in the determination of the bow wave /48, 49, 55, 50/. We obtained the corresponding formulas in Appendix IV by the stationary-phase method.

The problem of determining the junction between the constant and conical flows was also considered in /47/, where equations and particular solutions were obtained. However, the obtained solution does not satisfy all the conditions at B.

The results of this work are extended to conical flow around a wing in Appendix II, and to motions of a viscous and conducting fluid in Appendix IV.

The assumption of infinite conductivity results in an incorrectly posed problem; however, by introducing a finite conductivity and then passing to the limit it can be shown that the influence of a finite conductivity is only evident near the boundary. The work does not include problems dealt with in the candidate's thesis, which contains a solution of the linear case, and neither are the results of /27/ included.

The results of this investigation were presented at the Second All-Union Conference on Theoretical and Applied Mechanics, at a seminar on Hydromechanics under the direction of Academician Sedov at the Computer Center of the Siberian Branch of the Academy of Sciences of the USSR. Helpful comments were made, for which the author thanks the participants in the discussion. The author expresses his gratitude to Academician Ambartsumyan of the Academy of Sciences of the Armenian SSR, who contributed much to the preparation of this work and made a number of valuable remarks.

Formulation of the problem

Suppose an explosion occurs at a height h above the surface of an ideal compressible liquid, which occupies the lower half-space. The incident shock wave reaches the liquid surface at time t_2 and forms a reflected and a transmitted wave. Since the acoustic impedance $\rho_2 a_2$ of the liquid is considerably larger than $\rho_1 a_1$ for air, (ρ is the density, a is the velocity of sound), the shock wave is regarded as though reflected from a rigid barrier

and the pressure at the surface is assumed known. This is easily inferred from the refraction law for shocks, at least for the linear case /27/. If we denote the amplitudes of the incident and refracted waves by P_1 and P_2 respectively, then $\frac{P_2 - P_2^0}{P_1} \sim \frac{\rho_1 a_1}{\rho_2 a_2}$, where P_2^0 corresponds to reflection from a rigid half-space for which $\frac{\rho_1 a_1}{\rho_2 a_2} = 0$. Hence, when $\frac{\rho_1 a_1}{\rho_2 a_2}$ is small, so is the difference $P_2 - P_2^0$.

The order of magnitude of the energy flux in the second medium is given by $\frac{E_2}{E_1} \sim \frac{\rho_2 a_1}{\rho_2 a_2} \frac{1}{1 + \frac{\rho_1^2 a_1^2}{\rho_2^2 a_2^2}}$. Let the equation of state of the fluid have Tait's form /12/

$$P = B\left[\left(\frac{\rho}{\rho_0}\right)^n - 1\right], \qquad \text{(A)}$$

where P is the pressure, ρ is the density, ρ_0 is the initial density of the fluid, B and n are constants. For water $n = 7$, $B = 3{,}041$ kg/cm^2.

It can be easily shown that for pressures of up to 1,000 kg/cm^2, the characteristic parameter of the problem $\frac{P}{Bn}$ is small and a linear theory can be used to determine the pressure in the half-space.

If $\frac{\rho_1 a_1}{\rho_2 a_2} \sim \frac{P}{Bn}$, then the pressure given by the theory of reflection from a rigid half-space is valid in a first approximation; for $\frac{\rho_1 a_1}{\rho_2 a_2} \sim \left(\frac{P}{Bn}\right)^2$ the theory holds in a second approximation.

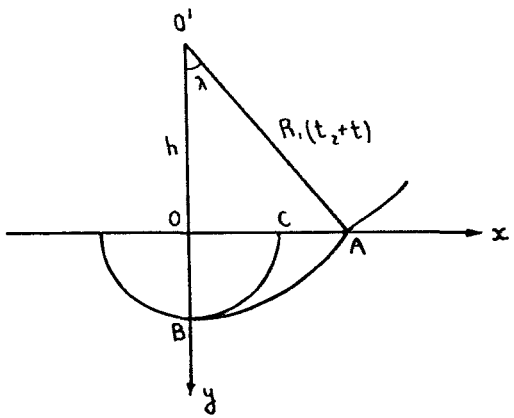


FIGURE 1.

When the explosion occurs at a point the problem has axial symmetry. The problem is two dimensional for an explosion along a straight line parallel to the bounding surface.

The disturbed motion is shown in Figure 1. The radius of the shock front is denoted by $R_1(t_2 + t)$, where t is the time after which the pressure

is transmitted into the fluid; the front is denoted by $R(t)$, where

$$R(t) = \sqrt{R_1^2(t_2+t) - h^2},\ h = R_1(t_2).$$

It is clear that the velocity $R'(t)$ is infinite at time $t=0$.

The $0x$ axis is taken along the undisturbed fluid surface, and the $0y$ axis points into the fluid. The pressure at the fluid boundary is assumed known in the following forms:

for the axisymmetric problem

$$P(x, y, t) = \begin{cases} P_1(x, t) & x < R(t) \\ 0 & x > R(t); \end{cases} \qquad \text{(B)}$$

for the two-dimensional problem

$$P(x, y, t) = \begin{cases} P_1(|x|, t) & |x| < R(t) \\ 0 & |x| > R(t), \end{cases} \qquad \text{(C)}$$

where $y = y(x, t)$.

Here $P_1(x, t)$ is the pressure according to the theory of reflection from a rigid half-space, and may include a correction factor /25/.

For the linear problem the boundary conditions are $y(x, t) = 0$ /27/. When the explosion takes place at the very surface, the motion is as depicted in Figures 2 and 3. The shock wave AB no longer completely covers the region of the initial conditions at the point 0 at the initial instant of time, and is bounded by the sound wave $CBB'C'$.

Figure 3 represents subsonic velocity over the surface given by $R'(t) < a$, where a is the velocity of sound in the fluid. The most interesting situation in practice is detonation, i.e., constant shock velocity over the surface ($R'(t) = V = \text{const}$) and constant pressure P_1^0 on the front at the point A. The boundary condition then becomes

$$P(x, y, t) = \begin{cases} P_1^0 P_a\left(\frac{x}{Vt}\right) & x < Vt \\ 0 & x > Vt \end{cases} \qquad \text{(D)}$$

for the axisymmetric problem and

$$P(x, y, t) = \begin{cases} P_1^0 P_a\left(\left|\frac{x}{Vt}\right|\right) \\ 0 \end{cases} \qquad \text{(E)}$$

for the two-dimensional one. In equations (B) and (C) the function $P_1(x, t)$ is equal to $P_\phi(t) f\left(\frac{x}{R}\right)$, where $f\left(\frac{x}{R}\right)$ and $P_a\left(\frac{x}{Vt}\right)$ are the pressure profiles behind the front at the surface, and $P_\phi(t)$ is the pressure on the front.

The equations of gas dynamics must be solved subject to a given boundary condition at the surface and the initial conditions $P = \frac{\partial P}{\partial t} = 0$ at $t = 0$.

Due to the existence of the small parameter $\frac{P_1^0}{Bn}$ or $\frac{P_\phi(t)}{Bn}$ the first or second approximation to the solution of the problem is sufficient, i.e., the fluid motion may be considered isentropic and irrotational /15/. The introduction of a potential enables the problem in the second approximation to be reduced to solving one equation for the velocity potential $\varphi(x, y, t)$.

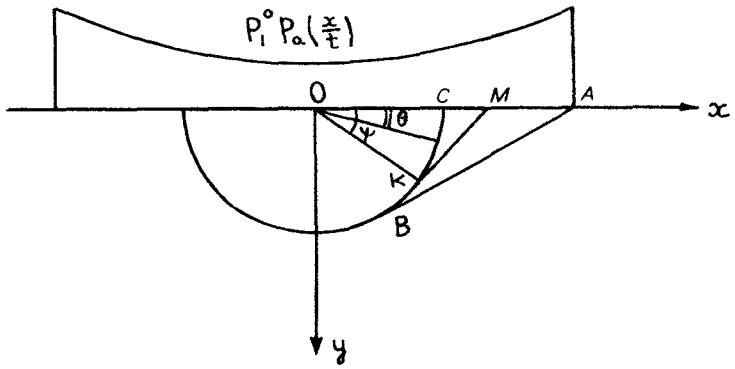


FIGURE 2.

A detailed examination of the boundary conditions will be given in the solution of specific problems. In this respect the most complicated problem will be that of the velocity transition R' through the velocity of sound a, when the shock breaks away from the immediate vicinity of A and irregular reflection takes place. If polar coordinates $x = r_1 \cos\theta$, $y = r_1 \sin\theta$, are introduced in the two-dimensional problem the equations of motion and continuity become

$$\begin{aligned}
&\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r_1} + v_\theta \frac{\partial v_r}{r_1 \partial \theta} - \frac{v_\theta^2}{r_1} = -\frac{1}{\rho}\frac{\partial P}{\partial r_1}, \\
&\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r_1} + v_\theta \frac{\partial v_\theta}{r_1 \partial \theta} + \frac{v_r v_\theta}{r_1} = -\frac{1}{\rho}\frac{\partial P}{r_1 \partial \theta}, \qquad \text{(F)} \\
&\frac{d\rho}{dt} + \rho\left(\frac{\partial v_r}{\partial r_1} + \frac{v_r}{r_1} + \frac{\partial v_\theta}{r_1 \partial \theta}\right) = 0,
\end{aligned}$$

where v_r and v_θ are the radial and transversal velocity components, respectively. In the axisymmetric problem the equations of motion are given by (F) if the second term within parenthesis in the continuity equation is replaced by $\frac{2v_r}{r_1} + \frac{v_\theta}{r_1}\operatorname{ctg}\theta$.

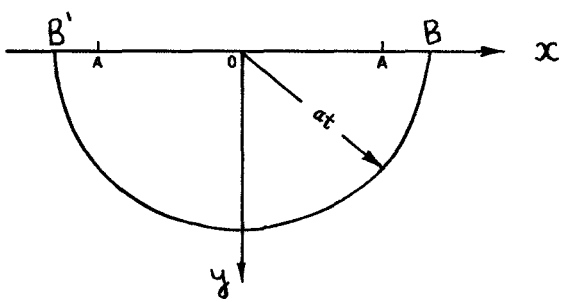


FIGURE 3.

In addition

$$\frac{d\rho}{dt} = \frac{1}{a^2}\frac{dP}{dt},$$

where a is the velocity of sound in the fluid. In the presence of a smali parameter, the problem may be solved to second order by neglecting the entropy variation on the shock front and assuming the motion to be potential. It is then fairly simple to obtain from (F) an equation for the potential $\varphi(r, \theta, t)$ in Lagrange's form:

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}\left(\frac{\partial\varphi}{\partial r_1}\right)^2 + \frac{1}{2}\left(\frac{\partial\varphi}{r_1\partial\theta}\right)^2 + \frac{a^2}{n-1} = \frac{a_0^2}{n-1}, \qquad (F)_1$$

$$v_r = \frac{\partial\varphi}{\partial r_1}, \quad v_\theta = \frac{\partial\psi}{r_1\partial\theta}, \quad \frac{a^2}{n-1} = \int\frac{dP}{\rho}$$

and in the two-dimensional problem /36/

$$\frac{\partial^2\varphi}{\partial t^2} + 2\frac{\partial\varphi}{\partial r_1}\frac{\partial^2\varphi}{\partial r_1\partial t} + 2\frac{1}{r_1^2}\frac{\partial\varphi}{\partial\theta}\frac{\partial^2\varphi}{\partial\theta\partial t} + \left(\frac{\partial\varphi}{\partial r_1}\right)^2\frac{\partial^2\varphi}{\partial r_1^2} +$$

$$+2\frac{1}{r_1^2}\frac{\partial\varphi}{\partial r_1}\frac{\partial\varphi}{\partial\theta}\frac{\partial^2\varphi}{\partial r_1\partial\theta} - \frac{1}{r_1^3}\frac{\partial\varphi}{\partial r_1}\left(\frac{\partial\varphi}{\partial\theta}\right)^2 + \frac{1}{r_1^2}\left(\frac{\partial\varphi}{\partial\theta}\right)^2\frac{\partial^2\varphi}{\partial\theta^2} - \qquad (F)'$$

$$-\frac{a^2}{r_1^2}\frac{\partial^2\varphi}{\partial\theta^2} - \frac{a^2}{r_1}\frac{\partial\varphi}{\partial r_1} = 0.$$

When the fluid is at rest ahead of the front, the conditions on the shock front expressing the continuity of mass and momentum become

$$\rho_0 D = \rho\ (D - v),\ P = \rho_0 D v, \qquad (G)$$

where D is the shock velocity and v is the velocity of the particles behind it. The continuity condition for the velocity component tangential to the front in potential flow can be replaced by $\varphi = 0$ or $\frac{\partial\varphi}{\partial t} = -Dv$. The equation of energy conservation /15/ on the shock front then agrees with the Lagrange form in $(F)_1$ (§ 2, Chapter I). The momentum equation gives $P = -\rho_0\frac{\partial\varphi}{\partial t}$ to second order, and the mass conservation equation gives Pfreim's formula for the shock velocity. The order of magnitude of the velocity $R'(t)$ over the surface should not differ from that of the velocity of sound a_0; for $\frac{R'(t)}{a_0} \ll 1$ the solution reduces to an acoustic dipole distribution (the problem is nonlinear near the boundary) as first pointed out by Grigoryan /42/.

Chapter I

FLUID PRESSURE FOR WAVE FRONT MOVING WITH CONSTANT VELOCITY OVER THE SURFACE

§ 1. Linear treatment of the two-dimensional problem

When $P_1^0 = \text{const}$ and $R'(t) = V = \text{const}$, the problem has similarity in time (self-similar problem) /14/ and its solution will depend on the parameters $\xi = \frac{x}{t}$, $\eta = \frac{y}{t}$.

In terms of the variables ξ and η the linearized equation for the potential $\tilde{\varphi} = \frac{\varphi}{t}$ is hyperbolic in the region ABC and elliptic in the region $CBB'C'$. In the region ABC the solution P will be constant along the characteristics MK (Figure 2), which are tangent to the circle $\sqrt{x^2 + y^2} = a_0 t$, a_0 being the initial velocity of sound.

In the region ABC, (E) takes the form

$$P = P_1^0 \, P_a\left(\frac{\xi_0'}{V}\right), \tag{1.1}$$

where ξ_0' is constant along the characteristics of the first family given by

$$\xi \cos\psi + \eta \sin\psi = a_0, \quad \cos\psi = \frac{a_0}{\xi_0'}. \tag{1.2}$$

Formulas (1.1) and (1.2) determine P in the whole region ABC. The velocity components v_x and v_y along the axes and the potential can be found from

$$P = -\rho_0 \frac{\partial \varphi}{\partial t}, \quad v_x = \frac{\partial \varphi}{\partial x}, \quad v_y = \frac{\partial \varphi}{\partial y}.$$

Using (1.2), we have

$$\varphi = -\frac{1}{\rho_0}\int_{\psi_*}^{\psi} P_1^0 P_a\left(\frac{\xi_0'}{V}\right) dt = -\frac{t}{\rho_0} P_1^0 \int_{\psi_*}^{\psi} P_a\left(\frac{a^0}{V\cos\psi}\right)(\xi \sin\psi + \eta\cos\psi)\, d\psi$$

for the potential φ, where ψ_0 is the value of ψ on the shock wave AB; in the linear case

$$\eta = \frac{V - \xi}{\sqrt{\frac{V^2}{a_0^2} - 1}}, \quad \cos\psi_0 = \frac{a_0}{V}. \tag{1.3}$$

When polar coordinates $x = r_1 \cos\theta$, $y = r_1 \sin\theta$, $r = \frac{r_1}{t}$ are introduced the

following expressions are derived:

$$v_r = \frac{1}{\rho_0 a_0} P_1^0 P_a\left(\frac{a_0}{V\cos\psi}\right)\cos(\psi-\theta) + \frac{1}{\rho_0 a_0}\int_{\psi_*}^{\psi} P_1^0 P_a\left(\frac{a_0}{V\cos\psi}\right)\sin(\psi-\theta)\,d\psi,$$

$$v_\theta = \frac{1}{\rho_0 a_0} P_1^0 P_a\left(\frac{a_0}{V\cos\psi}\right)\sin(\psi-\theta) - \frac{1}{\rho_0 a_0}\int_{\psi}^{\psi} P_1^0 P_a\left(\frac{a_0}{V\cos\psi}\right)\cos(\psi-\theta)\,d\psi,$$

$$\varphi = \frac{1}{\rho_0 a_0} r_1 \int_{\psi_*}^{\psi} P_1^0 P_a \sin(\psi-\theta)\,d\psi. \qquad (1.4)$$

Hence, one can determine the pressure distribution near the sonic line BC (Figure 2), $r = a_0$. Consider the pressure at the point (r, θ). Writing (1.2) in the form $r\cos(\psi-\theta) = a_0$ and taking into account the smallness of $\psi-\theta$, we have

$$r = a_0 + r_2,\ (a_0 + r_2)\cos(\psi-\theta) = a_0$$

or

$$r_2 = a_0\frac{(\psi-\theta)^2}{2},\ \psi-\theta = \sqrt{\frac{2(r-a_0)}{a_0}}. \qquad (1.5)$$

If (1.4) is considered near the curve $r = a_0$ and leading-order small quantities are retained, then

$$v_r = C(\theta)\,a_0 + B(\theta)\,a_0\sqrt{\frac{r}{a_0}-1}, \qquad (1.6)$$

$$v_\theta = C_1'(\theta)\,a_0, \qquad (1.7)$$

$$\bar{\varphi} = \frac{\varphi}{t} = \frac{1}{\rho_0 r_0}\int_{\psi_*}^{\theta} r\sin(\psi-\theta)\,P_a d\psi = C_1(\theta)\,r a_0, \qquad (1.8)$$

where

$$C(\theta) = \frac{P_1^0}{\rho_0 a_0^2} P_a\left(\frac{a_0}{V\cos\theta}\right) + \frac{P_1^0}{\rho_0 a_0^2}\int_{\psi_*}^{\theta} P_a\left(\frac{a_0}{V\cos\psi}\right)\sin(\psi-\theta)\,d\psi,$$

$$B(\theta) = \frac{P_1^0}{\rho_0 a_0^2} P_a'\left(\frac{a_0}{V\cos\theta}\right)\frac{a_0}{V}\frac{\sin\theta}{\cos\theta}\sqrt{2}.$$

The form (1.6) —(1.8) enables one to study the behavior of the solution near the fronts in the nonlinear problem.

Now consider the elliptic region in the linear formulation for which $V < a_0$, i.e., the subsonic case.

The disturbed region in Figure 3 is bounded by a curve, representing the vicinity of the sound wave $r_1 = a_0 t$, on which the pressure is equal to zero. It is known /16/ that the wave equation for the pressure can be reduced to a Laplace equation by means of a Chaplygin transformation. The new variables

$$\xi_1 = \frac{x}{a_0 t},\ \eta_1 = \frac{y}{a_0 t},\ \xi_1 = r'\cos\theta,\ \eta_1 = r'\sin\theta$$

can be introduced so that in terms of ε and θ, where

$$r' = \frac{2\varepsilon}{1+\varepsilon^2},$$

the pressure P is governed by a Laplace equation.

In terms of the variables $\xi' = \varepsilon \cos\theta$, $\eta' = \varepsilon \sin\theta$ the disturbed region will again be a semicircle of unit radius (Figure 4), where the section over which pressure is applied, $|\xi_1| \leqslant \frac{V}{a_0}$, corresponds to the segment $\eta'=0$,

$$|\xi'| \leqslant l, \; l = \frac{a_0}{V} - \sqrt{\frac{a_0^2}{V^2} - 1}.$$

The transformation $K = \frac{2z}{1+z^2}$, $z = \xi' + i\eta'$, $K = \xi_2^0 + i\eta_2$ maps the disturbed region onto the lower half plane. It is easily seen that the real-axis segment $\left[-\frac{V}{a_0}, \frac{V}{a_0}\right]$ in the plane ξ_2^0, η_2 corresponds to the section over which pressure is applied. The solution of the Laplace equation for the half plane is

$$P = \frac{1}{\pi}\eta_2 P_1^0 \int_{-\frac{V}{a_0}}^{\frac{V}{a_0}} P_a\left(\frac{|\xi_2|}{V/a_0}\right) \frac{d\xi_2}{(\xi_2 - \xi_2^0)^2 + \eta_2^2}. \tag{1.9}$$

The pressure variation with depth can be found for different forms of $P_a\left(\frac{|\xi_2|}{V/a_0}\right)$.

In contrast to the case $P_a = 1$, the pressure varies nonmonotonically with depth. When the pressure at the surface is propagated with a supersonic velocity $V > a_0$, (1.1) and (1.2) yield the solution in the hyperbolic region. The boundary conditions are then transferred to the arc BC of Figure 2 and the solution in the elliptic region is given by (1.9).

In the case $P_a = 1$, (1.9) takes the form

$$P = \frac{1}{\pi} P_1^0 \left(\operatorname{arctg} \frac{\frac{V}{a_0} - \xi_2^0}{\eta_2} + \operatorname{arctg} \frac{\frac{V}{a_0} + \xi_2^0}{\eta_2}\right), \tag{1.10}$$

which was derived in /17/. To determine the velocity components v_x and v_y when $V < a_0$ an analytic function $f = P + iS$ is introduced. Then from (1.9) we have (for $P_a = 1$) in the plane of z (Figure 4)

$$if = \frac{P_1^0}{\pi} \ln \frac{(z-l)\left(z - \frac{1}{l}\right)}{(z+l)\left(z + \frac{1}{l}\right)e^{i\pi}}. \tag{1.11}$$

The theory of conical flows /15/ yields that from a known function f one can construct v_x and v_y by the formulas (the role of z in the theory of conical flows is played here by t)

$$v_x + iv_y = \frac{1}{2\rho_0 a_0} \int z\,df + \frac{1}{z}\,d\bar{f},$$

where $\bar{f}$ is the conjugate function of f.

Hence, we obtain from (1.11)

$$-\frac{v_y}{\frac{P_1^0}{2\rho_0 a_0}\frac{1}{\pi}} = \left(l-\frac{1}{l}\right)\ln\left|\frac{z^2-l^2}{z^2-\frac{1}{l^2}}\right| + 2\left(l+\frac{1}{l}\right)\ln|z| + \left(l-\frac{1}{l}\right)\ln\frac{1}{l^2},$$

$$\frac{v_x}{\frac{P_1^0}{2\rho_0 a_0}\frac{1}{\pi}} = \left(l+\frac{1}{l}\right)\arg(z^2-l^2)\left(z^2-\frac{1}{l^2}\right) - 2\left(l+\frac{1}{l}\right)\arg(-z). \qquad (1.12)$$

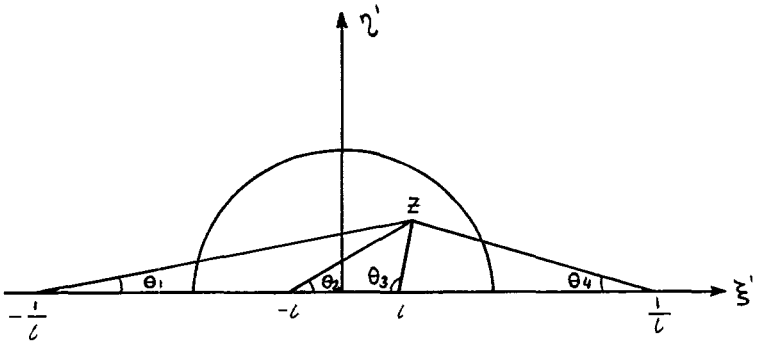


FIGURE 4.

Formulas (1.12) satisfy the condition that the solutions vanish on the unit circle $|z| = 1$.

Note that $\theta_1 = \arg\left(z+\frac{1}{l}\right)$, $\theta_2 = \arg(z+l)$, $\theta_3 = \pi - \arg(z-l)$, $\theta_4 = \pi - \arg\left(z-\frac{1}{l}\right)$, so that (1.10) can be rewritten as

$$P = -\frac{P_1^0}{\pi}(\theta_1+\theta_2+\theta_3+\theta_4-\pi). \qquad (1.13)$$

Formulas (1.12) are useful when solving the problem inside the disturbed region (Figure 3) in the second approximation /28,29/. When $V > a_0$, formula (1.12) takes a different form.

If $P_a = 1$ and $V > a_0$ it is easy to derive that in the region $BCC'B'$

$$P = \frac{1}{\pi} P_1^0 \arg \frac{1+z^2-2z\frac{a_0}{V}}{-1-z^2-2z\frac{a_0}{V}} \qquad (1.14)$$

and for v_x and v_y

$$-\frac{v_x}{\frac{1}{\rho_0 a_0}\frac{P_1^0}{\pi}} = \frac{a_0}{V}\arg\left(z^2-2z\frac{a_0}{V}+1\right)\left(z^2+2z\frac{a_0}{V}+1\right)\bar{z}^2,$$

$$\frac{v_y}{\frac{1}{\rho_0 a_0}\frac{P_1^0}{\pi}} = \frac{a_0}{V}\ln|\bar{z}|^2 - \sqrt{1-\frac{a_0^2}{V^2}}\arg\times$$

$$\times \frac{1-\left|\frac{z-a_0/V}{\mu}\right|^2+i\frac{z+\bar{z}-2a_0/V}{\mu}}{1-\left|\frac{z+a_0/V}{\mu}\right|^2+i\frac{z+\bar{z}+2a_0/V}{\mu}} + 2\theta_0\mu - \pi\mu, \qquad (1.15)$$

where $\theta_0 = \arccos a_0/V$, $\mu = \sin\theta$.

The behavior of the solution near the sonic line for the elliptic region will be considered in the solution of the nonlinear problem.

§ 2. Solution of the nonlinear problem by means of simple waves

When solving the two-dimensional problem in the second approximation use is made of the fact that the entropy change is of third order, and to second order the shock wave does not influence the flow behind it.

The solution of the equations of gas dynamics for a two-dimensional self-similar problem can be looked for in the form of simple waves /7, 19/, which represents a family of straight characteristics (the line MK in Figure 2) on which the parameters are constant:

$$\left.\begin{aligned}&(\xi - v_x)\cos\varphi + (\eta - v_y)\sin\varphi = a\\ &dv_x = \frac{dP}{\rho a}\cos\varphi\\ &dv_y = \frac{dP}{\rho a}\sin\varphi,\end{aligned}\right\} \qquad (2.1)$$

where $P = P(\xi')$, $v_x = v_x(\xi')$, $v_y = v_y(\xi')$ and $a = a(\xi')$ are constants along the lines (2.1) up to the surface of the fluid $\eta = \eta(\xi')$, $\xi = \xi'$. On this surface the boundary condition (E) holds, and for the angle between the normal to a characteristic and the $0x$ axis we have

$$\cos\varphi = \frac{a + v_x\cos\varphi + v_y\sin\varphi - \eta(\xi')\sin\varphi}{\xi'}.$$

The pressure is given by $P = P_1^0 P_a\left(\frac{\xi'}{V}\right) = P_1(\xi')$, and the equation of the surface is $\eta = \eta(\xi') = -\xi'\int_V^{\xi'}\frac{v_y}{\xi'^2}d\xi'$. It can be easily seen that when determining the shock wave, $\xi' \to V$ corresponds to $\eta(\xi') = 0$.

To determine the shock wave we should substitute the known parameters of motion in the formula for the shock velocity (Pfreim's formula /15/)

$$D = \frac{\eta - \xi\frac{d\eta}{d\xi}}{\sqrt{1+\left(\frac{d\eta}{d\xi}\right)^2}} = \frac{a + v + a_0}{2}, \qquad (2.2)$$

where according to (A) we have, in the first approximation, the velocity $a = a_0\left(1 + \frac{n-1}{2}\frac{P}{Bn}\right)$, $\rho = \rho_0\left(1 + \frac{P}{Bn}\right)$, $v = a_0\frac{P}{Bn}$.

All the quantities in (2.2) are taken in the first approximation, since ξ' must be determined along the shock wave to this degree of accuracy, whereas the pressure $P = P_1(\xi')$ is determined in the second approximation, where $P = P_1(V) + P_1'(V)(\xi' - V)$.

The quantity ξ' is not constant at the points where the characteristics (2.1) intersect the shock wave, but a function of the point on the shock itself. In the first approximation the equations of the characteristics (2.1) are near the shock wave

$$\eta = (\xi' - \xi)\ \mathrm{ctg}\varphi,$$
$$\cos\varphi = \frac{a_0\left(1 + \frac{n+1}{2}\frac{P}{Bn}\right)}{\xi'}; \qquad (2.1)'$$

$\xi_1' = V - \xi'$ is small, since the front is influenced by the characteristics in its immediate vicinity $\left(\xi_1' \sim \frac{P_1^0}{Bn}\right)$. Equations (2.1)' easily yield

$$\eta = \frac{V-\xi}{\sqrt{\frac{V^2}{a_0^2} - 1}} + \frac{\xi_1'\left(1 - \frac{V}{a_0^2}\xi\right)}{\left(\frac{V^2}{a_0^2} - 1\right)^{3/2}} + \frac{V-\xi}{\left(\frac{V^2}{a_0^2} - 1\right)^{3/2}}\frac{V^2}{a_0^2}\frac{n+1}{2}\frac{P_1^0}{Bn}$$

or

$$\eta = (V - \xi)\,\mathrm{tg}\alpha_1 + \frac{\xi_1'\left(1 - \frac{V}{a_0^2}\xi\right)}{\cos^3\alpha_1}\sin^3\alpha_1 + \frac{V-\xi}{\cos^3\alpha_1}\sin\alpha_1\frac{n+1}{2}\frac{P_1^0}{Bn}, \qquad (2.3)$$

where $\sin\alpha_1 = \frac{a_0}{V}$. When (2.3) is substituted in (2.2), the following expression is derived for ξ_1' along the shock AB (Figure 2):

$$\xi_1' = \frac{(V-\xi)\frac{n+1}{4}\frac{P_1^0}{Bn}}{\sin^2\alpha_1\left(1 - \frac{V}{a_0^2}\xi\right)} \qquad (2.4)$$

and the equation of the shock wave in the form

$$\eta = (V - \xi)\,\mathrm{tg}\alpha_1 + \frac{n+1}{4}\frac{P_1^0}{Bn}(V-\xi)\frac{\sin\alpha_1}{\cos^3\alpha_1}, \qquad (2.5)$$

where $P_1(V) = P_1^0$. These represent the shock wave and the pressure on it in the second approximation. To substantiate the obtained solution, we note that the equations of gas dynamics have a solution similar to that of (2.1) /7/, and therefore it is sufficient to verify that the conditions on the shock wave are satisfied. This we do now.

The motion here is potential. Consider the energy equation in the Lagrange form

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}v^2 + \int\frac{dP}{\rho} = \left(\frac{\partial\Phi}{\partial t}\right)_1 + \left(\frac{1}{2}v^2\right)_1 + \left(\int\frac{dP}{\rho}\right)_1. \qquad (2.6)$$

where Φ is the potential of the motion and the subscript 1 refers to some point in the fluid. Since there is no fluid motion ahead of the front, the

continuity of the tangential velocity component $\frac{\partial \Phi}{\partial s}$ (this continuity is shown later) implies that $\frac{\partial \Phi}{\partial s}$, and consequently Φ, vanish along the front. Therefore, $\frac{\partial \Phi}{\partial t} + Dv = 0$ along the front, where v here coincides with the velocity component normal to the shock wave. Substituting this information in (2.6) we have

$$-Dv + \frac{1}{2} v^2 + \int \frac{dP}{\rho} = -(Dv_1)_1 + \left(\frac{1}{2} v^2\right)_1 + \left(\frac{dP}{\rho}\right)_1. \tag{2.7}$$

On the other hand, the energy equation in a shock transition is /12/

$$\frac{1}{2}(v - D)^2 + \int \frac{dP}{\rho} = \frac{1}{2} D^2 + \left(\int \frac{dP}{\rho}\right)_0, \tag{2.8}$$

where the subscript 0 denotes the state of the fluid at rest.

If $\frac{1}{2} D^2$ is transferred to the left-hand side, the quantities without subscripts in equations (2.7) and (2.8) are identical. Then in the sense of the Lagrange representation one can replace the subscript 0 by 1 and relations (2.7) and (2.8) agree.

Passing to the remaining relations on the shock we note that small quantities up to and including the second order are considered. It is now shown that the remaining relations on the shock wave can be obtained from system (2.1). If the velocity is given by its components

$$v_x = v \cos \beta, \quad v_y = v \sin \beta,$$

then (2.1) may be written in the form

$$dv = \frac{dP}{\rho a} \cos(\varphi - \beta),$$
$$v d\beta = \frac{dP}{\rho a} \sin(\varphi - \beta). \tag{2.9}$$

In the linear problem the velocity is directed along the normal to the characteristic coinciding with the shock wave, and so $\varphi_0 - \beta_0 = 0$, $(\varphi = \varphi_0, \beta = \beta_0)$. Therefore $\varphi - \beta$ is small, and

$$dv = \frac{dP}{\rho a},$$
$$v d\beta = \frac{dP}{\rho a}(\varphi - \beta), \tag{2.10}$$

up to third-order small quantities. The undisturbed state ahead of the shock front is defined by $v = 0$, $P = 0$ and $\rho = \rho_0$, as well as the parameter $\frac{P_1}{Bn}$ characterizing the smallness of the motion. Equations (2.10) yield

$$\left(\frac{dv}{dP}\right)_{P=0} = \frac{1}{\rho_0 a_0}; \quad \left(\frac{d^2 v}{dP^2}\right)_{P=0} = -\frac{n+1}{2} \frac{1}{\rho_0 a_0} \frac{1}{Bn},$$

so that v is obtained in the form

$$v = \frac{P}{\rho_0 a_0} - \frac{n+1}{4} \frac{P}{\rho_0 a_0} \frac{P}{\rho_0 a_0^2}. \tag{2.11}$$

Equation (A) expanded up to the third order, is

$$\frac{\rho}{\rho_0} = 1 + \frac{1}{n}\frac{P}{B} - \frac{1}{n}\frac{n-1}{n}\frac{1}{2}\left(\frac{P}{B}\right)^2. \tag{2.12}$$

This corresponds tc the result derived from the second-order equation of energy or the shock adiabat /12/ for the fluid.

The continuity and momentum equations are

$$\rho_0 D = \rho\,(D - v),$$
$$P = \rho_0 D v, \tag{2.13}$$

where D is the shock velocity.

The approximate relation

$$P\frac{P}{Bn}\left(1 - \frac{n+1}{2}\frac{P}{Bn}\right) = \rho_0 v^2$$

or

$$v = a_0\frac{P}{Bn}\left(1 - \frac{n+1}{4}\frac{P}{Bn}\right)$$

can be obtained from (2.12) and (2.13), and agrees with (2.11) since $Bn = \rho_0 a_0^2$. The equation of the characteristic, (2.3), gives to first order

$$\frac{d\eta}{d\xi} = -\operatorname{tg}\alpha_1 - \frac{\sin\alpha_1}{\cos^3\alpha_1}\frac{n+1}{2}\frac{P_1^0}{Bn} - \frac{\sin^2\alpha_1}{\cos^3\alpha_1}\frac{\xi_1'}{a_0}. \tag{2.14}$$

Thus, along the characteristic $\xi_1' = \text{const}$ the angle of inclination to Ox, $\alpha = \alpha_1 + \alpha_2$, is governed by

$$-\operatorname{tg}\alpha_1 - \frac{1}{\cos^2\alpha_1}\alpha_2 = \frac{d\eta}{d\xi}.$$

A comparison with (2.14) shows that the angle at which the shock wave is inclined to Ox, determined from (2.5) in the form

$$\alpha_1 + \frac{n+1}{4}\frac{P_1^0}{Bn}\operatorname{tg}\alpha_1, \tag{2.15}$$

is not the arithmetic mean of the angles of inclination of the characteristics α_1 in front of the shock and α behind. This simple wave is thus distinguished from the well-known Riemann wave, although the solution is correct to second order. The difference is due to the third term on the right-hand side of (2.14). It was shown that the solution of (2.1) satisfies all the conditions on the shock wave to second order, except that for the fluid velocity component tangential to the shock, v_s. We shall now prove that $v_s = O\left(\frac{P_1^0}{Bn}\right)^3$.

The expression for the velocity component tangential to the shock wave has the form

$$v_s = v\sin(\lambda - \beta),$$

where λ is the angle between the normal to the shock and the Ox axis. In a linear approximation $\lambda_0 = \beta_0$, $\beta_0 = \frac{\pi}{2} - \alpha_1$, so that v_s is at least a second-order

small quantity. It can be easily seen that the pressure on the front AB is constant to first order and equal to P_1^0. The increments dP and da along the shock wave are therefore second-order small quantities. It follows from the second equation of (2.10) that $d\beta = 0$ to first order. If we retain first-order small quantities in the expansion of $a\ (x,\ y,\ t)$ then

$$a\ (x,\ y,\ t) = a_0\left(1 + \frac{n-1}{2}\frac{P_1^0}{Bn}\right) + \cdots,$$

where $a_0\left(1 + \frac{n-1}{2}\frac{P_1^0}{Bn}\right)$ is a constant, and the omitted terms are small quantities of second order. Thus, da is a second-order small quantity.

The velocity component tangential to the shock is

$$v_x \sin\lambda - v_y \cos\lambda,$$

where λ is the angle between Ox and the normal to the shock.

Hence, (2.1) gives the following relation:

$$v_x \sin\lambda - v_y \cos\lambda = \int \frac{2}{n-1}\, da \sin(\lambda - \varphi), \qquad \text{where} \qquad \frac{2da}{n-1} = \frac{dP}{\rho a}.$$

Since φ and λ are almost identical in the linear approximation, $\lambda - \varphi$ is a first-order small quantity and da is second order. Consequently, $v_x \sin\lambda - v_y \cos\lambda$ is a small quantity of third order, which was to be proved.

In the above proof we considered a narrow region near the shock wave, whereas the integration limits for φ are not rigorously defined (φ_0 and φ can be taken for the limits, where φ_0 is the value of φ at the point A). We therefore present another proof which corresponds better to the problem.

The tangential velocity component at the front is

$$v \sin(\lambda - \beta).$$

The solution (2.5) of the problem yields

$$-\operatorname{tg}\lambda = \frac{d\eta}{d\xi} = -\operatorname{tg}\alpha_1 - \frac{n+1}{4}\frac{\sin\alpha_1}{\cos^3\alpha_1}\frac{P_1^0}{Bn}$$

for the shock, where α_1 is the angle of the characteristic for the linear problem, namely,

$$\sin\alpha_1 = \frac{a_0}{V}.$$

If λ is expressed as $\lambda = \lambda_1 + \lambda_2 \frac{P_1^0}{Bn}$, the above equation gives

$$\lambda_1 = \alpha_1,\ \lambda_2 = \frac{n+1}{4}\operatorname{tg}\alpha_1.$$

Thus, λ is constant up to second-order small quantities. From the second equation of (2.10) we have $v d\beta \sim da \sin(\varphi - \beta)$, or $d\beta = 0$ to the accuracy that $da = 0$. This equality takes place on the whole shock, and therefore both λ and β are constant up to and including first-order small quantities. The boundary condition for β, as for all the parameters, is set at the point $x = Vt,\ y = 0$, but at this point it is a necessary condition on the shock that $\beta = \lambda$. Since β and λ are constant up to the first order inclusive, $\beta - \lambda$ is a small quantity of the second order along the front. The tangential velocity

component at the wave front is then $v \sin(\lambda - \beta)$, a small quantity of third order. Another fact should be mentioned. When solving the problem in the second approximation it is sufficient to find the quantities β, φ, D in the first approximation. In fact, the shock front velocity, from equations (2.13), is given by

$$D^2 = \frac{P}{\rho_0\left(1 - \frac{\rho_0}{\rho}\right)}.$$

If $\frac{\rho}{\rho_0}$ is expanded up to the second order inclusive, the following expression is derived for D:

$$D = a_0\left(1 + \frac{n+1}{4}\frac{P}{Bn}\right), \tag{2.16}$$

i.e., the first approximation. In order to determine second-order small quantities in the expression for D, one must include terms of the order of $\left(\frac{P}{Bn}\right)^3$ in the expansion of $\frac{\rho}{\rho_0}$, i.e., depart from the isentropic approximation. This is also pointed out in /26/ in another connection.

Thus, when determining the physical parameters v_x, v_y, P, ρ and v in the second approximation, it is sufficient to take the quantities φ, D and β to first order; in fact, $v_x = v\cos\beta$, $v_y = v\sin\beta$ and so first-order accuracy for β provides second-order accuracy for v_x and v_y.

One can also determine the shock in the first approximation without using (2.1).

For the characteristics of the first family we have an equation to first order /27/

$$\frac{\eta - \xi\frac{d\eta}{d\xi}}{\sqrt{1 + \left(\frac{d\eta}{d\xi}\right)^2}} = a + v. \tag{2.17}$$

Represent the characteristics by parameters ξ', where $\xi = \xi'$, $\eta = 0$ is the point of intersection of the characteristic $\eta(\xi, \xi')$ with Ox. The front AB is influenced by characteristics for which $\xi_1' = V - \xi'$ is small. If $\eta = \eta_0 + \eta_2$ is substituted in the differential equation of the characteristics, we obtain that $\eta = \eta_0$ is given by (2.17) in the linear case ($a + v = a_0$) with boundary conditions $\xi = \xi', \eta = 0$, where $\eta_0 = \frac{\xi' - \xi}{\sqrt{\frac{\xi'^2}{a_0^2} - 1}}$, and η_2, from (2.17), is to first order given by

$$\eta_2 - (\xi - a_0\sin\alpha_1)\frac{d\eta_2}{d\xi} = P_1^0\frac{a_0}{\cos\alpha_1}\frac{n+1}{2}. \tag{2.18}$$

The solution of this equation subject to the condition $\xi = \xi', \eta = 0$ is, to first order,

$$\eta_2 = \frac{P_1^0}{Bn}\frac{a_0}{\cos\alpha_1}\frac{n+1}{2}\frac{V - \xi}{V - a_0\sin\alpha_1}, \tag{2.19}$$

where we set $\xi' = V$.

When $\eta = \eta_0 + \eta_2$ is substituted in the condition (2.2) on the shock wave, where in the expression for η_0 first-order small quantities in ξ_1' are retained, formula (2.4) for ξ_1' and the shock wave equation (2.5) are obtained.

We note that the characteristics (2.1) have the envelope

$$\xi - v_x = a\cos\varphi - (a' + v_x'\cos\varphi + v_y'\sin\varphi)\sin\varphi,$$

$$\eta - v_y = a\sin\varphi + (a' + v_x'\cos\varphi + v_y'\sin\varphi)\cos\varphi.$$

(Primes denote differentiation with respect to the parameter φ.) Hence,

$$(\xi - v_x)^2 + (\eta - v_y)^2 = a^2 + (a' + v_x'\cos\varphi + v_y'\sin\varphi)^2. \qquad (2.20)$$

The envelope of the characteristics is thus formed in front of the parabolic line in the variables ξ, η for the exact equation governing the potential:

$$(\xi - v_x)^2 + (\eta - v_y)^2 = a^2.$$

On the curve (2.20) the acceleration $\sim \dfrac{\partial P}{\partial t}$ and tends to infinity. The solution behind the limiting line near the wave $r_1 = a_0 t$ is given in § 3, where for a constant flow in ABC it follows from (3.36) that the flow behind the parabolic line is elliptic, a fact also pointed out in /21/. A consideration of simple waves is also given by Parker (Bodies which Adjoin Regions of Simple Wave Supersonic Flow. — Quart. J. Mech. Appl. Math., Vol. 18. 1965), who takes into account the shock waves AB and the wave in front of BC. In addition, the author's view that in the case of a compressive flow near the front edge the influence of the vertex 0 extends to the whole simple wave up to the point A, seems to us incorrect (Appendix II). The case of pressure propagation corresponds to the flow around a wing of lenticular shape, i.e., to a rarefied flow, considered in the indicated paper. A general consideration of simple waves is given in a paper on the theory of conical flows by Bulakh (Prikl.Mat.Mekh., Vol. 18. 1954). His paper, however, contains many unclear points. In particular, it is stated on page 400 that at points of the envelope S the acceleration $F_{\eta\eta}$ changes sign. In the proof of this statement, the parameters at points L and N of a simple wave situated on different sides of S are considered equal, whereas the points L and N of the unique solution are situated on different lines. On page 404, in proving that the envelope should pass through the parabolic point, it is shown that $|k'(o)| > 2(M_1^2-1)^{1/4}$, where $k(\xi)$ is the slope of the straight characteristics along the curved characteristic $\eta(\xi)$. The Mach number is M_1, and $\xi = 0$ are the parabolic points. It follows from Figure 4 that $k'(o) < 0$, and therefore $k'(o) < -2(M_1^2 - 1)^{1/4}$. This inequality is obtained from the relation $\eta''(o) = -2(M_1^2 - 1)^{1/4}$, which is found on page 399, derived from the condition $k'(o) = 0$. This contradicts the foregoing. When $k'(o) < -2(M_1^2-1)^{1/4}$ for $\eta'(o)$, it is obtained on page 398 that $\eta''(o) > 0$ and the whole proof does not hold. The statement that the acceleration vanishes on the straight line passing through the parabolic point of a simple wave is thus not proved. However, in our case (as in most cases known to us), the simple wave does not reach the parabolic line, and the indicated consideration is not required.

We note that (2.20) is a limiting line; in §3 it will be shown, however, that to second order a shock does not form in front of it. This apparently requires further consideration. In determining the boundary conditions in the second approximation for the case $V < a_0$ on the shock wave $r = a_0$, Moore /28/ and Lighthill /26/ do not take Lighthill's solution itself /3/, but the first term of its expansion at the exit near the frontal region, in accordance with the method of inner and outer expansions (Van Dyke, Perturbation Methods in Fluid Mechanics: Applied Mathematics and Mechanics, Vol.8. 1964). Bulakh /21/ obtains Lighthill's solution /3/ by the same method. Bulakh's solution for the velocities agrees with Lighthill's at the discontinuity, but he states that the solution /3/ for the potential is not exact.

§ 3. Pressure near the front of a sound wave

The formulas obtained in § 1 for the linear case are not valid near the front of a sound wave, since they give infinite accelerations. Such cases were studied by Lighthill /3/. Lighthill's asymptotic method makes it possible to find a solution on the shock wave in the second approximation when the linear solution has a branch in the form of a square root near the sonic line.

To obtain the solution near the shock wave, we use Whitham's approach /4/ for solving problems in the first approximation and obtain results which were earlier obtained by Lighthill's technique /36/. The complete equivalence of both methods is shown for the two-dimensional problem, but the method in /4/ is simpler; this is particularly so for an inhomogeneous conducting fluid (§ 9, Chapter II and Appendix IV). Then we show that the solution obtained by the above method satisfies the equations in the second approximation, which is true both for the two-dimensional and the axisymmetric problems (§ 6).

The pressure near the sonic line $r = a_0$ can be determined at its boundary with the undisturbed fluid from formula (1.9) in § 1, as well as from the more general formula for the three-dimensional case /27/. When $V < a_0$ this boundary is the curve BB' (Figure 3); in the case $V > a_0$ it is the segment of BB' (Figure 2).

Up to leading-order small quantities we have /27/

$$\frac{P(x, y, t)}{P_1^0} = f(\theta)\frac{\sqrt{t - \frac{r_1}{a_0}}}{\sqrt{\frac{r_1}{a_0}}}, \tag{3.1}$$

where

$$f(\theta) = \frac{1}{\pi}\sin\theta\sqrt{2}\,\frac{V}{a_0}\int_{-1}^{1}\frac{P_a\left(\left|\frac{\xi}{V}\right|\right)d\xi}{\left(1 - \frac{V}{a_0^2}\frac{x\xi}{t}\right)^2}.$$

Near $r = a_0$ the velocity potential has the form

$$\frac{\varphi}{t} = -\gamma a_0^2\,\frac{2}{3}f(\theta)\left(1 - \frac{r_1}{a_0 t}\right)^{3/2},\ \gamma = \frac{P_1^0}{\rho_0 a_0^2}. \tag{3.1'}$$

In the case $P_a = 1$

$$f(\theta) = \frac{2}{\pi}\sin\theta\,\frac{\sqrt{2}\,M}{1 - M^2\cos^2\theta},\quad M = \frac{V}{a_0}. \tag{3.1''}$$

The pressure on the wave BB' in Figure 3 for $M > 1$ and for the wave BB' in Figure 2 is obtained by replacing the linear characteristic variables $t - \frac{r_1}{a_0}$ by improved ones in the linear solution /4/.

Let $y_1 = const$ be the equation of the characteristics. Replacing $t - \frac{r_1}{a_0}$ by y_1 in (3.1) and leaving second-order small quantities in the exact equation of the characteristics

$$1 - v_{r_1}\frac{\partial t}{\partial r_1} - v_\theta\frac{\partial t}{r_1\partial\theta} = a\sqrt{\left(\frac{\partial t}{\partial r_1}\right)^2 + \left(\frac{\partial t}{r_1\partial\theta}\right)^2}, \tag{3.2}$$

we obtain a differential equation for the characteristics which have the same form as one-dimensional ones, i.e.,

$$\frac{\partial t}{\partial r_1} = \frac{1}{a_0} - \frac{1}{a_0}\frac{n+1}{2}\frac{P}{Bn}$$

or, after substituting (3.1) and integrating,

$$t = \alpha(r_1) - (\beta(r_1) - \beta(a_0 y_1))\sqrt{y_1} + y_1, \tag{3.3}$$

where $\alpha = \frac{r_1}{a_0}$, $\beta(r_1) = \frac{1}{\sqrt{a_0}}\frac{n+1}{2}\frac{P_1^0}{Bn}\sqrt{r_1}\,2f(\theta)$, and y_1 is a small typical time t. By virtue of the smallness of y_1 the term $\beta(a_0 y_1)$ in (3.3) may be neglected for finite t.

Using the formula for the shock wave velocity

$$\frac{t}{r_1} = \frac{1}{a_0} - \frac{1}{a_0}\frac{n+1}{4}\frac{P}{Bn}, \tag{3.4}$$

a relation between $\sqrt{y_1}$ and the coordinate r_1 of the points on the shock can be derived from (3.3) and (3.4) in the form

$$\sqrt{y_1} = \frac{3}{4}\beta. \tag{3.5}$$

From (3.5) and (3.1), where $t - \frac{r_1}{a_0}$ is replaced by y_1, the pressure at the shock is

$$\frac{P}{P_1^0} = \frac{3}{2}\frac{n+1}{2} f^2(\theta)\frac{P_1^0}{\rho_0 a_0^2} \tag{3.6}$$

and the shock front equation (according to (3.4)) is

$$r_1 = a_0 t + a_0 t\,\frac{3}{16}\,(n+1)^2 f^2(\theta)\left(\frac{P_1^0}{Bn}\right)^2. \tag{3.7}$$

With the aid of (3.3), the pressure distribution behind the shock wave is

$$\sqrt{y_1} = \frac{\beta + \sqrt{\beta^2 + 4\left(t - \frac{r_1}{a_0}\right)}}{2}$$

$$\frac{P}{P_1^0} = f(\theta)\sqrt{\frac{y_1}{\frac{r_1}{a_0}}}. \tag{3.8}$$

On the wave $t = \frac{r_1}{a_0}$ we have $\sqrt{y_1} = \beta$ and

$$\frac{P}{P_1^0} = (n+1)\left(\frac{P_1^0}{Bn}\right) f^2(\theta). \tag{3.9}$$

Let us find the curve on which the accelerations tend to infinity (the dashed curve in Figure 5).

From (3.1) we obtain for $\frac{\partial P}{\partial t}$

$$\frac{\partial P}{\partial t} = P_1^0 f(\theta)\frac{1}{2\sqrt{t}}\frac{1}{\sqrt{y_1}}\frac{\partial y_1}{\partial t},$$

and from (3.3)

$$\frac{\partial y_1}{\partial t} = \frac{1}{1-\beta(r_1)\dfrac{1}{2\sqrt{y_1}}}. \tag{3.9'}$$

The condition $\frac{\partial y_1}{dt} = \infty$ gives

$$\sqrt{y_1} = \frac{1}{2}\beta(r_1)$$

and so the limiting-line equation is

$$t = \frac{r_1}{a_0} - \frac{1}{4}\beta^2(r_1), \tag{3.10}$$

which is obviously situated ahead of the shock wave outside the disturbed region.

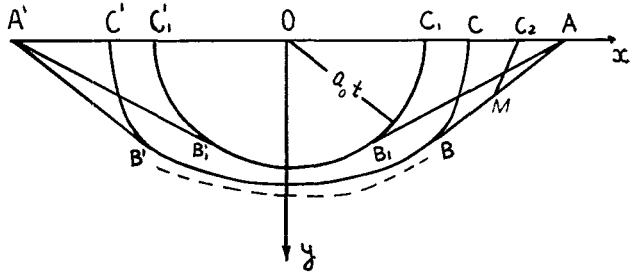


FIGURE 5.

The pressure distribution behind the shock increases monotonically towards the fluid boundary $y=0$. If $t - \frac{r_1}{a_0} \sim \left(\frac{P_1}{Bn}\right)^{\alpha}$, $\alpha < 1$, one may assume $\beta = 0$ in (3.8), i.e., the solution (3.8) becomes linear.

Since the linear solution was used to derive (3.8), it can be easily shown that the conditions on the shock wave are satisfied in the second approximation. The same result will now be obtained by Lighthill's technique /26, 3/.

Equations (F) govern the two-dimensional problem expressed in terms of the variables $r = \frac{r_1}{t}$ and θ for the potential $\frac{\varphi}{t} = \bar{\varphi}$, and may be written as

$$\frac{\partial^2\bar{\varphi}}{\partial r^2}\left(\left(\frac{\partial\bar{\varphi}}{\partial r} - r\right)^2 - a^2\right) + \frac{1}{r^3}\left(2r - \frac{\partial\bar{\varphi}}{\partial r}\right)\left(\frac{\partial\bar{\varphi}}{\partial\theta}\right)^2 - 2\frac{1}{r^2}\frac{\partial\bar{\varphi}}{\partial\theta}\left(r - \right.$$

$$\left. - \frac{\partial\bar{\varphi}}{\partial r}\right)\frac{\partial^2\bar{\varphi}}{\partial r\partial\theta} + \frac{1}{r^4}\left(\left(\frac{\partial\bar{\varphi}}{\partial\theta}\right)^2 - r^2a^2\right)\frac{\partial^2\bar{\varphi}}{\partial\theta^2} - \frac{a^2}{r}\frac{\partial\bar{\varphi}}{\partial r} = 0,$$

$$a^2 = a_0^2 - (n-1)\frac{1}{2}\left(\frac{\partial\bar{\varphi}}{\partial r}\right)^2 - (n-1)\frac{1}{2}\left(\frac{\partial\bar{\varphi}}{r\partial\theta}\right)^2 - (n-1)\left(\bar{\varphi} - r\frac{\partial\bar{\varphi}}{\partial r}\right). \tag{3.11}$$

Suppose $\bar{\varphi}$ and r have the form $\bar{\varphi} = \bar{\varphi}_0 + \bar{\varphi}_1$, $r = R + r_1(\theta) + r_2(\theta)$, where $\bar{\varphi}_0$, r_1 are first-order small quantities, and $\bar{\varphi}_1$, r_2 are second-order small quantities in $\gamma = \frac{P_1^0}{Bn}$. The equation for the second approximation up to $\bar{\varphi}_1(R, \theta)$ obtained from (3.11) is

$$\frac{\partial^2 \bar{\varphi}_1}{\partial R^2}(R^2 - a_0^2) - a_0^2 \frac{1}{R^2}\frac{\partial^2 \bar{\varphi}_1}{\partial \theta^2} - \frac{a_0^2}{R}\frac{\partial \bar{\varphi}_1}{\partial R} = -\frac{\partial^2 \bar{\varphi}_0}{\partial R^2}\Big(2Rr_1 -$$

$$-(n+1)R\frac{\partial \bar{\varphi}_0}{\partial R} + (n-1)\bar{\varphi}_0 + 2Rr_2 + r_1^2 - (n+1)R\frac{\partial \bar{\varphi}_1}{\partial R} +$$

$$+(n-1)\bar{\varphi}_1 + \Big(r_1 - \frac{\partial \bar{\varphi}_0}{\partial R}\Big)^2 + \frac{n-1}{2}\Big(\frac{\partial \varphi_0}{\partial R}\Big)^2 + \frac{n-1}{2}\Big(\frac{\partial \bar{\varphi}_0}{R\partial\theta}\Big)^2 -$$

$$-r_1\frac{\partial \bar{\varphi}_0}{\partial R}(n-1)\Big) - \frac{2}{R^2}\Big(\frac{\partial \bar{\varphi}_0}{\partial \theta}\Big)^2 + \frac{2}{R}\frac{\partial \bar{\varphi}_0}{\partial \theta}\frac{\partial^2 \bar{\varphi}_0}{\partial \theta \partial R} + \frac{(n-1)R\frac{\partial \bar{\varphi}_0}{\partial R} - (n-1)\bar{\varphi}_0 + 2r_1\frac{a_0^2}{R}}{R^2}\frac{\partial^2 \bar{\varphi}_0}{\partial \theta^2} -$$

$$-\frac{a_0^2}{R^2}\,2\,\frac{\partial r_1}{\partial \theta}\frac{\partial^2 \bar{\varphi}_0}{\partial \theta \partial R} - \frac{a_0^2}{R^2}\frac{\partial^2 r_1}{\partial \theta^2}\frac{\partial \bar{\varphi}_0}{\partial R} + \frac{(n-1)R\frac{\partial \bar{\varphi}_0}{\partial R} - (n-1)\bar{\varphi}_0 - \frac{a_0^2}{R}r_1}{R}\frac{\partial \bar{\varphi}_0}{\partial R}. \quad (3.12)$$

The linear solution $\bar{\varphi}_0(R, \theta)$ is given by (3.1').

In the case $V < a_0$ near BB' in Figure 3 or $V > a_0$ near BB' in Figure 2, $\frac{\partial^2 \bar{\varphi}_0}{\partial R^2}$ has a singularity according to (3.1'). Equating the terms in parenthesis multiplying $\frac{\partial^2 \bar{\varphi}_0}{\partial R^2}$ to zero for $R = a_0$, and sorting out first and second orders, we obtain

$$r_1 = 0, \quad r_2 = \frac{n+1}{2}\frac{\partial \bar{\varphi}_1}{\partial R}. \quad (3.13)$$

If R tends to a_0 in the equation for $\bar{\varphi}_1$, then to second order

$$\frac{\partial \bar{\varphi}_1}{\partial R} = -(n+1)\lim_{R \to a_0}\frac{\partial^2 \bar{\varphi}_0}{\partial R^2}\frac{\partial \bar{\varphi}_0}{\partial R}$$

or, from (3.1'),

$$\frac{\partial \bar{\varphi}_1}{\partial R} = \gamma^2 f^2(\theta)\frac{n+1}{2}a_0. \quad (3.14)$$

The momentum condition at the shock front /36/ is

$$\frac{\partial \tilde{\varphi}_0}{\partial R} + \frac{\partial \bar{\varphi}_1}{\partial R} = \frac{4}{n+1}(M - a_0), \quad M = R + r_2,$$

and when (3.13), (3.14) and (3.1') are used, where $\frac{r}{t} = R$, the following relations are derived:

$$M = a_0 + a_0 \frac{3}{16}(n+1)^2 \gamma^2 f^2(\theta), \quad P = \frac{3}{4}\rho_0 a_0^2 (n+1)^2 \gamma^2 f^2(\theta),$$

which agree with (3.6). After the above operations, the right-hand side of equation (3.12) is independent of r_2, i.e., the equations are identical in r and R. The boundary-value problem for $\tilde{\varphi}_1$ in the region should be solved in the variables R, θ subject to the boundary condition

$$P = \frac{1}{2}\rho_0 a_0^2 (n+1)^2 \gamma^2 \rho^2$$

on $R = a_0$, obtained from (3.8).

If in the expression

$$\frac{\partial \bar{\varphi}}{\partial r} = \frac{\partial \tilde{\varphi}_0}{\partial R} + \frac{\partial \tilde{\varphi}_1}{\partial R}$$

we substitute $\frac{\partial \bar{\varphi}_0}{\partial R}$ from (3.1) with r replaced by R, and $\frac{\partial \bar{\varphi}_1}{\partial R}$ from (3.14), then to second order near BB'

$$\frac{\partial \bar{\varphi}}{\partial r} = a_0 \gamma f(\theta) \frac{\sqrt{t - \frac{r_1}{a_0} + \frac{r_2}{a_0}}}{\sqrt{\frac{r_1}{a_0}}} + \gamma^2 f^2(\theta) a_0 \frac{n+1}{2}, \tag{3.15}$$

where $\frac{r_2}{a_0} = \left(\frac{n+1}{2}\right)^2 \gamma^2 f^2(\theta)$. However $P = \rho_0 a_0 \frac{\partial \bar{\varphi}}{\partial r}$, so that expression (3.8) for P is easily derived from (3.15). The first term on the right-hand side of (3.15) for $r_1 = a_0 t$ gives, in contrast to the linear result (3.1), the nonzero value

$$\frac{\partial \tilde{\varphi}}{\partial r} = a_0 \gamma^2 f^2(\theta) \frac{n+1}{2}, \tag{3.16}$$

which agrees with the value of $\frac{\partial \tilde{\varphi}}{\partial r}$ on the limiting line (3.10). Lighthill's principle /26/ for obtaining a more valid solution is to take the linear solution (3.1) not on the linear sound wave $r_1 = a_0 t$, but on the limiting line (3.10), and add to it the value $\frac{\partial \tilde{\varphi}_1}{\partial R}$, obtained from equation (3.12) for $R = a_0$ or on the limiting line. For arbitrary $R(t)$, $P_\Phi(t)$ and a finite $R'(0)$ the solution near BB' is again given by (3.1), (3.3), where V and P_1^0 are replaced by $R'(0)$ and $P_\Phi(0)$. In order to show that for finite times the nonlinear solution in the vicinity of BB' is entirely determined by (3.1), the variables $\bar{\tau}(r_1, \theta, t) = t\bar{\varphi}(\xi, \theta, t)$, $\xi = \frac{r_1}{t}$ are introduced in the equation (F)' with polar coordinates $x = r_1 \cos\theta$, $y = r_1 \sin\theta$ for the potential in the two-dimensional problem. The equation

$$2t\frac{\partial \bar{\varphi}}{\partial t} - 2\xi t \frac{\partial^2 \bar{\varphi}}{\partial \xi \partial t} + \frac{\partial^2 \bar{\varphi}}{\partial \xi^2}\left(\xi - \frac{\partial \bar{\varphi}}{\partial \xi}\right)^2 - \frac{\partial^2 \bar{\varphi}}{\partial \xi^2} a^2 - a^2 \frac{1}{\xi}\frac{\partial \tilde{\varphi}}{a\xi} + 2\frac{\partial \tilde{\varphi}}{\partial \xi} t \frac{\partial^2 \tilde{\varphi}}{\partial \xi \partial t} = 0$$

is obtained when the leading terms are retained. If the notation

$$\tilde{\varphi} = \tilde{\varphi}_0 + \tilde{\varphi}_1, \quad \xi = R + r_1 + r_2$$

is introduced according to Lighthill's technique, then with the condition $R = a_0$, $r_1 = 0$, an equation for $\frac{\partial \tilde{\varphi}_1}{\partial R}$ is derived in the form

$$2a_0 t \frac{\partial}{\partial t}\left(\frac{\partial \tilde{\varphi}_1}{\partial R}\right) + (n-1)\lim_{R \to a_0} \frac{\partial^2 \tilde{\varphi}_0}{\partial R^2} \frac{\partial \tilde{\varphi}_0}{\partial R} + a_0 \frac{\partial \tilde{\varphi}_1}{\partial R} = 0,$$

with the solution

$$\sqrt{t}\,\frac{\partial \tilde{\varphi}_1}{\partial R} = \frac{n+1}{2}\gamma^2 f^2(\theta)\sqrt{t}\,a_0 + C.$$

From the condition that $\frac{\partial \tilde{\varphi}_1}{\partial R}$ is finite at $t = 0$ it follows that $C = 0$. Hence $\frac{\partial \tilde{\varphi}_1}{\partial R} = \frac{n+1}{2}\gamma^2 a_0 f^2(\theta)$, which again has the form (3.15). For the case $V < a_0$ (Figure 3) one can consider the pressure variation on BB' when $P_a = 1$. From (3.1'') for $f'(\theta) = 0$ we have $f'(\theta) = \frac{1 - M^2 - M^2 \sin^2\theta}{(1 - M^2\cos^2\theta)^2}\cos\theta$ with limiting values

$$\theta_1 = \frac{\pi}{2}, \quad \theta_{2,3} = \pm \arcsin \frac{\sqrt{1 - M^2}}{M}$$

where $\theta_{2,3}$ will only be real for $1 < 2M^2$.

Consider the range of variation $0 \leqslant \theta \leqslant \frac{\pi}{2}$. For the second derivative we have

$$f''\left(\frac{\pi}{2}\right) = -1 + 2M^2, \quad f''(\theta_{2,3}) = \frac{-M^2 2\sin\theta\cos\theta}{(1 - M^2\cos^2\theta)^2}\cos\theta < 0.$$

Thus, when $M < \frac{1}{\sqrt{2}}$ the pressure rises from the fluid surface along the shock wave up to a maximum value

$$\frac{P}{P_1^0} = \frac{3}{4}(n+1)^2 \frac{P_1^0}{Bn}\left(\frac{2\sqrt{2}}{\pi} M\right)^2.$$

For $M > \frac{1}{\sqrt{2}}$ the maximum point corresponds to $\theta = \theta_2$, and the pressure later falls to its value at $\theta = \frac{\pi}{2}$. Near the free surface, according to formula (3.6), the pressure is of higher order and may be taken equal to zero.

Consider the behavior of the solution near the parabolic line BC (Figure 5), having the equation $r = a + v_r$.

The solution of the linear problem near B_1C_1 was found in § 1, formula (1.9). In the case $P_a = 1$ and near $r = a$ for $r < a_0$,

$$P = P_1^0 - \frac{P_1^0}{\pi} \frac{2\sqrt{2}\,M\sqrt{t - \frac{r_1}{a_0}}\sin\theta}{(\cos^2\theta\, M^2 - 1)\sqrt{\frac{r_1}{a_0}}}. \tag{3.17}$$

Since the pressure in the region ABC is equal to P_1^0, we may conclude from (3.17) that behind BC there is rarefaction.

For an arbitrary boundary function $P_a\left(\frac{\xi}{V/a_0}\right)$ in front of CB, the solution near the wave $r=a_0$ is given by (1.6), (1.7), (1.8). Behind BC one can use (1.9) to represent the functions P, v_r, v_θ in the forms

$$\frac{\varphi}{a_0^2 t} = \frac{\tilde{\varphi}}{a_0^2} = C_1(\theta) + C(\theta)(r_1' - 1),$$

$$\frac{1}{a_0} v_r = C(\theta) + B(\theta)(r_1' - 1)^{\frac{1}{2}}, \quad r_1' > 1,$$

$$\frac{1}{a_0} v_r = C(\theta) + A(\theta)(1 - r_1')^{\frac{1}{2}}, \quad r_1' < 1, \tag{3.18}$$

where $r_1' = \frac{r_1}{a_0 t}$, $C_1(\theta)$, $C(\theta)$, and $B(\theta)$ are found from (1.6)—(1.8), and $A(\theta)$ from (1.9). In particular, for the linear distribution $P_a = A_1 \xi/M + B_1$, $M = \frac{V}{a_0}$, $A_1 + B_1 = 1$, we have

$$A(\theta) = -\frac{1}{\pi} P_a \frac{P_1^0}{Bn} \frac{\sin\theta\, 2M\sqrt{2}}{M^2\cos^2\theta - 1} +$$

$$+ \frac{1}{\pi}\frac{A_1}{M}\sin\theta\frac{P_1^0}{Bn}\frac{\sqrt{2}\ln(M^2\cos^2\theta - 1)}{\cos^2\theta} - \frac{1}{\pi}\frac{A_1 2\sin\theta\frac{P_1^0}{Bn}\sqrt{2}}{\cos^2\theta\left(M + \frac{1}{\cos\theta}\right)}. \tag{3.19}$$

Thus near BC we have expressions (3.18).

Note that $B(\theta) > 0$, $A(\theta) < 0$ always, i.e., there is rarefaction behind BC and a shock wave does not form. This result is obtained from the form of $\frac{\partial P}{\partial r}$ near $r_1 = a_0 t$.

The absence of a shock near BC follows from Lighthill's technique /3/, which enables one to find the strength of the shock wave and also the solution near the parabolic line in the second approximation. This method is described now for our nonstationary problem and generalized, bearing in mind the more general applications of Chapter II.

Consider the equation for the velocity potential $f = \frac{\varphi}{a_0^2 t}$ in the nonlinear case. The equation for $f = \frac{\tilde{\varphi}}{a_0^2}$ has the form /36/

$$f_{rr}\frac{a_0^2(f_r - r)^2 - a^2}{a_0^2} + 2\frac{1}{r^3}\left(r - \frac{1}{2}f_r\right)f_\theta^2 -$$

$$- \frac{2}{r^2}f_\theta(r - f_r)f_{r\theta} + \frac{1}{r^4}\left(f_\theta^2 - r^2\frac{a^2}{a_0^2}\right)f_{\theta\theta} - \frac{a^2}{a_0^2}\frac{1}{r}f_r = 0, \tag{3.20}$$

where

$$r = r_1', \quad \frac{a^2}{a_0^2} = 1 - \frac{n-1}{2}f_r^2 - \frac{n-1}{2}\frac{1}{r^2}f_\theta^2 - (n-1)(f - rf_r).$$

(Subscripts denote differentiation.)

According to /3/, the variables r and f are represented in the forms

$$r = R + r_1(\theta) + r_2(\theta),$$
$$f = f_1(R, \theta) + f_2(R, \theta), \tag{3.21}$$

where $r_1(\theta), f_1$ and $r_2(\theta), f_2$ are first- and second-order small quantities respectively with respect to $\frac{P_1^0}{Bn} = \gamma$. Substituting (3.21) in (3.20) and equating to zero the terms corresponding to the first power of γ in the coefficient of f_{RR} we find that

$$r_1 = -\frac{n-1}{2} C_1(\theta) + \frac{n+1}{2} C(\theta). \tag{3.22}$$

As R tends to 1 in (3.20)

$$\lim_{R \to 1} \{(n+1) f_{1R}^0 f_{1RR}\} + (f_{2R} + f_{200})_{R=1} -$$
$$- \{r_1(\theta) + (n-1)(C_1 - C)\} C(\theta) = 0, \; f_{1R}^0 = f_{1R} - C(\theta).$$

According to (3.18), either the expression $\frac{1}{2}(n+1) B^2(\theta)$ or $-\frac{1}{2}(n+1) A^2(\theta)$ has a limit, depending from which direction we approach BC (from without or within). The difference in the values of the quantities within and without is denoted by Δ. If the coefficient of f_{RR} to second order is then equated to zero, where $f_{00} = f_{RR} r_1'^2(\theta) + \cdots$, we obtain

$$\Delta f_{2R} = \frac{1}{2}(n+1)(A^2(\theta) + B^2(\theta)); \; 2r_2 - (n+1) f_{2R} + (n-1) f_2 - r_1'^2(\theta) = 0. \tag{3.23}$$

The difference in the values behind and ahead of BC is

$$\Delta f_{2R} = \frac{1}{2}(n+1)(A^2(\theta) + B^2(\theta)); \; \Delta r_2 = \frac{1}{4}(n+1)^2 (A^2(\theta) + B^2(\theta)). \tag{3.24}$$

Suppose $r = \eta(\theta) = 1 + \eta_1 + \eta_2$ on the shock wave; then

$$R + r_1 + r_2 = 1 + \eta_1 + \eta_2, \; r_1 = \eta_1, \; R - 1 = \eta_2 - r_2.$$

Introduce the notation

$$\eta_2 - r_2 = \frac{1}{4}(n+1)^2 (F(0) + B^2(\theta)) \quad (R > 1),$$
$$\frac{1}{4}(n+1)^2 (F(\theta) - A^2(\theta)) \quad (R < 1). \tag{3.25}$$

Lighthill /3/ showed that in the obtained expressions one may set $C_1(\theta) = C(\theta) = C_1'(\theta) = 0$ without affecting the calculated intensity S of the shock wave. The discontinuity condition gives a relation for the normal velocity component $v_n a_0$ and the shock velocity D /3/, where

$$-\Delta v_n = \frac{2}{n+1}\left(\frac{a^2}{D - v_n a_0} - D + v_n a_0\right).$$

Since

$$v_n = v_r \cos(n, r) + v_\theta \cos(n, \theta) = \frac{f_r - \frac{1}{\eta} f(\theta) \frac{1}{\eta} \eta'(\theta)}{\sqrt{1 + \frac{1}{\eta^2} \eta'^2(\theta)}},$$

which in our approximation gives f_r, we have

$$\Delta v_n = \Delta f \sqrt{1 + \frac{1}{\gamma_i^2}\gamma_i'^2(\vartheta)}\,.$$

With the aid of formula (3.11) for a^2 and the formula for D, the relation

$$D = a_0 \frac{\gamma_i(\vartheta)}{\sqrt{1 + \frac{1}{\gamma_i^2}\gamma_i'^2(\vartheta)}}$$

is derived. Retaining $\gamma_i'^2(\vartheta)$, which is second order in γ, the following connection is obtained on the discontinuity:

$$\Delta f_R = \frac{2}{n+1}(n-1)f - 2f_r - \frac{2}{n+1}\gamma_i'^2(\vartheta) + \frac{2}{n+1}2(\gamma_i - 1)$$

or

$$\Delta f_R = \frac{2}{n+1}(n-1)f_2 - 2f_{2R} - 2f_{1R} - \frac{2}{n+1}\gamma_i'^2(\vartheta) + \frac{2}{n+1}2\gamma_{i2}. \qquad (3.26)$$

Hence,

$$\Delta f_R = \Delta f_{1\sigma} + \Delta f_{2R} = A(\vartheta)\frac{1}{2}(n+1)(A^2(\vartheta) - F(\vartheta))^{\frac{1}{2}} -$$

$$- B(\vartheta)\frac{1}{2}(n+1)(B^2(\vartheta) + F)^{\frac{1}{2}} + \frac{1}{2}(n+1)(A^2(\vartheta) + B^2(\vartheta)). \qquad (3.27)$$

By substituting the value outside BC for f_{1R} equation (3.26) yields

$$A(\vartheta)\frac{n+1}{2}(A^2(\vartheta) - F(\vartheta))^{\frac{1}{2}} - B(\vartheta)\frac{n+1}{2}(B^2(\vartheta) + F(\vartheta))^{\frac{1}{2}} + \frac{1}{2}(n+1)(A^2(\vartheta) + B^2(\vartheta)) =$$

$$= -(n+1)B(\vartheta)(B^2(\vartheta) + F(\vartheta))^{\frac{1}{2}} + \frac{2}{n+1}\frac{1}{2}(n+1)^2(B^2(\vartheta) + F(\vartheta))$$

or

$$A(\vartheta)(A^2(\vartheta) - F(\vartheta))^{\frac{1}{2}} + B(\vartheta)(B^2(\vartheta) + F(\vartheta))^{\frac{1}{2}} = A^2 - B^2 = 2F(\vartheta).$$

The pressure on the shock is /3/

$$\Delta P = \rho_c a_0^2 \Delta f_r = \rho_0 a_0^2 (n+1)\{(A^2(\vartheta) - F(\vartheta))^{\frac{1}{2}} A(\vartheta) -$$

$$- B(\vartheta)(B^2(\vartheta) + F(\vartheta))^{\frac{1}{2}} + A^2(\vartheta) + B^2(\vartheta)\} =$$

$$= S(\vartheta)\rho_0 a_0^2 (n+1) - (n+1)\rho_0 a_0^2 \{F(\vartheta) + B^2 - B^2(B^2 + F)^{\frac{1}{2}}\}. \qquad (3.28)$$

Hence, S is given by

$$S = F + B^2 - B\sqrt{B^2 + F} = A\sqrt{A^2 - F} + A^2 - F,$$

$$F = S - \frac{1}{2}B^2 + \frac{1}{2}B\sqrt{B^2 + 4S},$$

$$16S^4 - 24(A^2 + B^2)S^3 + 9(A^2 + B^2)S^2 - 4(A^2 + B^2)A^2B^2S = 0. \qquad (3.29)$$

This equation was investigated by Lighthill /3/. In our case $A < 0$, $B \geqslant 0$, and we have a solution $S = 0$. This corresponds to the absence of a shock wave.

The position BC in the first approximation is

$$\eta(\theta) = 1 + r_1^0(\theta),$$

where $r_1^0(\theta)$ is given by (3.22).

We now obtain (3.29) by the method given in /4/. The equation of the characteristics is written according to (3.2), (2.6), (3.18) as follows:

$$\frac{\partial r_1}{\partial t} - v_r + v_\theta \frac{\partial r}{r\partial\theta} = a\sqrt{1+\left(\frac{\partial r}{r\partial\theta}\right)^2}, \tag{3.30}$$

$$a = a_0 - \frac{n-1}{4}\frac{1}{a_0}v_r^2 - \frac{n-1}{4}\frac{1}{a_0}v_\theta^2 - \frac{n-1}{2}\frac{\partial\varphi}{\partial t}\frac{1}{a_0} + \frac{1}{8}\frac{(n-1)^2}{a_0^3}\left(\frac{\partial\varphi}{\partial t}\right)^2,$$

$$\frac{\partial\varphi}{\partial t} = \tilde{\varphi} - r\frac{\partial\tilde{\varphi}}{\partial r} = \begin{cases} -\frac{P_1'}{\rho_0} - a_0^2 A(\theta)\left(1-\frac{r_1}{a_0 t}\right)^{\frac{1}{2}}, & r_1 < a_0 t \\ -\frac{P_1'}{\rho_0} - a_0^2 B(\theta)\left(\frac{r'}{a_0 t}-1\right)^{\frac{1}{2}}, & r' > a_0 t \end{cases}$$

$$P_1' = P_1^0 P_n\left(\frac{a_0}{V\cos\theta}\right)$$

Equation (3.30) can be written up to second-order small quantities in the form

$$\frac{\partial r_1}{\partial t} - \frac{a_0}{2}\left(\frac{1}{r}\frac{\partial r}{\partial\theta}\right)^2 = v_r + a - v_\theta\frac{\partial r}{r\partial\theta}.$$

The expressions for v_r, v_θ, and $\frac{\partial\varphi}{\partial t}$ are substituted from (3.18), and the linear characteristic variables replaced by the exact ones, i.e., $t - \frac{r_1}{a_0} \to y_1$, $\frac{r'}{a_0} - t \to y_1$. The equations of the characteristics are then given by

$$\frac{\partial r_1}{\partial t} = a_0\frac{1}{2}r_1^{0'2}(\theta) + a_0 C(\theta) + a_0 A(\theta)\sqrt{\frac{y_1}{t}} - C_1'(\theta)r_1^{0'}(\theta)a_0 +$$

$$+ a_0 - \frac{n-1}{4}a_0 C'^2(\theta) - \frac{n-1}{4}a_0 C_1'^2(\theta) + \frac{n-1}{2}\frac{P_1'}{\rho_0 a_0} +$$

$$+ \frac{n-1}{2}A(\theta)a_0\sqrt{\frac{y_1}{t}} + \frac{1}{8}\frac{(n-1)^2}{a_0^3}\left(\frac{P_1'}{\rho_0}\right)^2, \quad r_1 < a_0 t$$

and

$$\frac{\partial r'}{\partial t} = a_0\frac{1}{2}r_1^{0'2}(\theta) + a_0 C(\theta) + a_0 B(\theta)\sqrt{\frac{y_1'}{t}} -$$

$$- C_1'(\theta)r_1^{0'}(\theta)a_0 + a_0 \quad \frac{n-1}{4}a_0 C'^2(\theta) - \frac{n-1}{4}a_0 C_1'^2(\theta) +$$

$$+ \frac{n-1}{2}\frac{P_1'}{\rho_0 a_0} + \frac{n-1}{2}B(\theta)a_0\sqrt{\frac{y_1'}{t}} + \frac{1}{8}\frac{(n-1)^2}{a_0^3}\left(\frac{P_1'}{\rho_0}\right)^2, r' > a_0 t.$$

Their solutions are

$$r_1 = a_0 t + a_0\frac{n+1}{2}A(\theta)2\sqrt{t}\sqrt{y_1} + C_3(\theta)ta_0 - a_0 y_1, r_1 < a_0 t,$$

$$r' = a_0 t + a_0\frac{n+1}{2}B(\theta)2\sqrt{t}\sqrt{y_1'} + C_3(\theta)ta_0 + a_0 y_1', r' > a_0 t;$$

$$C_3(\theta) = r_1^{0'2}(\theta) + C(\theta) - C_1'(\theta)r_1^{0'}(\theta) - \frac{n-1}{4}C'^2(\theta) -$$

$$- \frac{n-1}{4}C_1'^2(\theta) + \frac{n-1}{2}\frac{P_1'}{\rho_0 a_0^2} + \frac{1}{8}\frac{(n-1)^2}{a_0^4}\left(\frac{P_1'}{\rho_0}\right)^2. \tag{3.31}$$

The shock wave equation becomes

$$\frac{\partial r_1}{\partial t} - \frac{1}{2} a_0 \left(\frac{1}{r} \frac{\partial r}{\partial \theta} \right)^2 = \frac{a + v_n + a_1 + v_{n1}}{2} + \frac{1}{8} \frac{(v_n + a - v_{n1} - a_1)^2}{a_0},$$

where the subscript 1 refers to the region $r_1 > a_0 t$. Since $v_n = v_r - v_\theta \frac{1}{r} \frac{\partial r}{\partial \theta}$, the following is true to second order:

$$\eta(\theta) = 1 + \frac{1}{2} r_1^{0\prime 2}(\theta) + C(\theta) - C_1'(\theta) r_1^{0\prime}(\theta) + \frac{n-1}{2} \frac{P_1'}{\rho_0 a_1^2} +$$

$$+ \frac{n+1}{2} \frac{A(\theta) \sqrt{\frac{y_1}{t}} + B(\theta) \sqrt{\frac{y_1'}{t}}}{2} - \frac{n-1}{4} C'^2(\theta) - \frac{n-1}{4} C_1'^2(\theta) + \frac{1}{8} \frac{(n-1)^2}{a_0^2} \left(\frac{P_1'}{\rho_0} \right)^2 =$$

$$= 1 + C_3(\theta) + \frac{n+1}{2} \frac{A \sqrt{\frac{y_1}{t}} + B \sqrt{\frac{y_1'}{t}}}{2}.$$

When the equations of the characteristics are equated on the shock wave the relationships

$$r_1 = r' = a_0 t \eta(\theta),$$

$$\frac{n+1}{2} A(\theta) 2\sqrt{t} \sqrt{y_1} - y_1 = \frac{n+1}{2} B(\theta) 2\sqrt{t} \sqrt{y_1'} + y_1 = \frac{n+1}{4} (A \sqrt{y_1} \sqrt{t} + B \sqrt{y_1'} \sqrt{t}),$$

are derived. In terms of the notation $\sqrt{\frac{y_1}{t}} = t_1 \frac{n+1}{2}$, $\sqrt{\frac{y_1'}{t}} = t_2 \frac{n+1}{2}$, this takes the form

$$A(\theta) 2t_1 - t_1^2 = B(\theta) 2t_2 + t_2^2,$$

$$A(\theta) 2t_1 - t_1^2 = \frac{A(\theta) t_1 + B(\theta) t_2}{2}. \tag{3.32}$$

With the additional notation

$$A(\theta) t_1 - B(\theta) t_2 = S_1, \quad S_1 = 2S,$$

we have

$$t_2 = \frac{3A(\theta) t_1 - 2t_1^2}{B(\theta)} = \frac{A(\theta) t_1 - S_1}{B(\theta)},$$

$$2At_1 - 2t_1^2 = -S_1. \tag{3.33}$$

Similarly

$$t_1 = \frac{3Bt_2 + 2t_2^2}{A} = \frac{Bt_2 + S_1}{A},$$

$$2Bt_2 + 2t_2^2 = S_1,$$

hence yielding the result

$$-2At_1 + 2t_1^2 = 2Bt_2 + 2t_2^2. \tag{3.34}$$

It follows from (3.32) that

$$t_1 = \sqrt{A^2 - F} + A, \quad t_2 = \sqrt{B^2 + F} - B,$$

so that (3.34) may be written as

$$A\sqrt{A^2-F}+B\sqrt{B^2-F}+A^2-B^2=2F,$$

which agrees with (3.29). The first equation of (3.32) is satisfied identically. Since there is no shock wave in the case $A<0$, $B>0$ /3/, we have a continuous transition through the parabolic line $\frac{r_1}{a_0 t}=1+r_1^2$ (see (3.22)), where $y_1=y_1'=0$. For simplicity consider the case $P_a=1$. It can then be easily seen that the values a of the velocity of sound, taken from (A) in the form $\frac{a}{a_0}=1+\frac{n-1}{2}\frac{P}{Bn}+\frac{1-n^2}{8n^2}\left(\frac{P}{B}\right)^2$ and from (3.30), agree to second order according to (3.31). If we replace $\frac{x}{t}$ and $\frac{y}{t}$ by x and y, and the velocity components in the constant-flow region ABC by U_x and U_y, a system of coordinates can be introduced which move with velocity

$$U_x,\ U_y:\ x_1=x-U_x,\ y_1=y-U_y. \tag{3.35}$$

With the introduction of polar coordinates r, θ and r_1, θ_1, we have to second order $r^2=r_1^2+2rU_r-(v_r^2+U_\theta^2)$, $r=r_1+U_r-\frac{U_\theta^2}{2r}$. The velocity of sound in ABC is

$$a_1=a_0+\frac{n-1}{2}\frac{P_1}{\rho_0 a_0}+\frac{1-n^2}{8}\left(\frac{P_1}{\rho_0 a_0^2}\right)^2 a_0,$$

and so the first line of (3.31) reduces to

$$r_1=a_1+a_1\frac{n+1}{2}A\,2\sqrt{y_1}-a_0 y_1, \tag{3.36}$$

where $\frac{y_1}{t}$ is replaced by y_1. If small quantities are discarded, the values of r, θ, and r_1, θ_1 agree. Consequently, the linear solution in (3.18) can be taken as v_{r_1} and v_{θ_1}, where $v_{r_1}\approx v_r-a_0 C$, $v_{\theta_1}\approx v_\theta-a_0 C_1'$. Expression (3.18) then yields (3.36) and

$$\frac{v_{r_1}}{a_0}=A\sqrt{y_1},\ \frac{v_{\theta_1}}{a_0}=\frac{A'}{r_1}\int_{a_0}^{r_1}\sqrt{y_1}\,dr+\frac{A}{r}\int_{a_0}^{r_1}\frac{1}{2\sqrt{y_1}}\frac{\partial y_1}{\partial\theta}\,dr, \tag{3.37}$$

where to second order one can set $\theta=\theta_1$.

The solution (3.36), (3.37) now agrees with (3.8), giving, however, rarefaction. It is therefore correct to second order for $\sqrt{y_1}=0\left(\frac{P_1}{\rho_0 a_0^2}\right)$. To verify it, P can be eliminated from equations (F) and the resulting equations written to second order in the variables x_1, y_1, v_{x_1}, v_{y_1} or in the polar coordinates r_1, θ_1:

$$\frac{\partial v_{r_1}}{r_1\partial\theta_1}=v_{\theta_1}+\frac{1}{r_1}\frac{\partial v_\theta}{\partial r_1},\ (r_1-v_{r_1})^2\frac{\partial v_{r_1}}{\partial r_1}-a^2\frac{\partial v_{r_1}}{\partial r_1}-$$
$$-(r_1-v_{r_1})\left(\frac{\partial v_{r_1}}{r_1\partial\theta_1}v_{\theta_1}-\frac{v_{\theta_1}^2}{r_1}\right)-v_{\theta_1}r_1\frac{\partial v_{\theta_1}}{\partial r_1}-a^2\frac{\partial v_{\theta_1}}{r_1\partial\theta_1}-a^2\frac{v_{r_1}}{r_1}=0. \tag{3.38}$$

It is easily seen that the first equation is satisfied by the solution (3.18) or (3.37). In the second equation the terms containing $U_r=Ca_0$, $v_{\theta_2}=C_1'a_0$ cancel; in the variables r_1, θ_1 they simply equal zero and a one-dimensional equation with respect to r_1 is obtained. It will be shown in § 6 that (3.36) is a

solution of (3.38). In the case under consideration $a^2 = a_1^2 + (n-1)\,a_1 v_{r_1}$; this is pointed out in /21/.

§ 4. Solution near the point of contact of the regions and in the whole region

Consider the problem near the point B (Figure 5), at which the elliptic and hyperbolic regions meet on the shock wave. In the linear case this corresponds to the point B_1 (Figure 5), whose polar angle is given by the equation $\cos\theta_0 = \frac{a_0}{V}$. The solution of § 3 holds for a bounded function $f(\theta)$, but from (3.1) it follows that $f(\theta)$ is unbounded near the point $\theta = \theta_0$ on the shock wave. The solution (3.6) on the shock holds as long as the difference $\theta - \theta_0$ is of higher order than $\left(\frac{P_1^0}{Bn}\right)^{\frac{1}{2}}$, namely

$$\theta - \theta_0 \sim \left(\frac{P_1^0}{Bn}\right)^{\alpha},\ \alpha < \frac{1}{2}.$$

In solution (3.6) the order of the pressure is $\left(\frac{P_1^0}{Bn}\right)^{2-2\alpha}$, i.e., the pressure is given in this approximation. It can be easily shown that as θ tends to the point θ_0 along the parabolas

$$\theta - \theta_0 = C\left(\frac{y_1}{t}\right)^{\alpha_1},\ \alpha_1 < \frac{1}{2}$$

the same order is maintained for the pressure.

The solution for $\theta - \theta_0 \sim \left(\frac{P_1^0}{Bn}\right)^{\alpha}$, $\alpha < \frac{1}{2}$ is (3.6), where simplifications are possible, e.g., when $P_a = 1$ near $\theta = \theta_0$ we have

$$f(\theta) = \frac{f_1(\theta_0)}{\theta - \theta_0} = \frac{\frac{1}{\pi}\sqrt{2}}{\theta - \theta_0} \tag{4.1}$$

Using the relationship $P_a(1) = 1$ in the integrand of (3.1) and the fact that $\frac{x}{a_0 t} = \cos\theta$ approximately, it can be easily shown that (4.1) holds for an arbitrary function $P_a\left(\frac{V}{a_0}\right)$.

The solution along the shock wave can thus be continued toward the point $\theta = \theta_0$ up to a distance of the order of

$$\theta - \theta_0 \sim \left(\frac{P_1^0}{Bn}\right)^{\alpha},\ \alpha < \frac{1}{2}.$$

The condition $\alpha < \frac{1}{2}$ is introduced so that terms containing $\frac{\partial t}{\partial \theta}$ in equation (3.2) for the characteristics will be of a higher order and may be discarded. The condition that equations (3.2) be one-dimensional (no derivatives with respect to θ) is

$$\frac{v_\theta}{\theta - \theta_0} \ll \frac{P}{\rho_0 a_0}. \tag{4.2}$$

When $\theta - \theta_0 \sim \left(\frac{P_1^0}{Bn}\right)^{\alpha}$, we have in virtue of (3.6)

$$\frac{P}{Bn} \sim \left(\frac{P_1^0}{Bn}\right)^{2-2\alpha}, \quad v_\theta \sim \frac{\bar{\varphi}}{\theta - \theta_0} \sim \frac{y_1 \frac{P}{Bn}}{\theta - \theta_0},$$

and in virtue of (3.5)

$$\frac{v_\theta}{\theta - \theta_0} \sim \frac{y_1 \frac{P}{Bn}}{(\theta - \theta_0)^2} \sim \left(\frac{P_1^0}{Bn}\right)^{2-4\alpha}.$$

The condition $v_\theta \ll \frac{P}{\rho_0 a_0}(\theta - \theta_0)$ thus leads to

$$2 - 4\alpha > 0, \ \alpha < \frac{1}{2}.$$

For $\alpha = \frac{1}{2}$ the relationship $v_\theta \sim \frac{P}{\rho_0 a_0}(\theta - \theta_0)$ is derived, where derivatives with respect to θ cannot be neglected in equation (3.2). The flow in the vicinity of B will be two dimensional (dependent on r and θ).

The solutions of § 3, given by formula (3.8), are second-order small quantities near the shock front and possess finite derivatives.

In fact, from (3.9') we have

$$\frac{\partial P}{\partial t} = P_1^0 f(\theta) \frac{1}{2\sqrt{t}\sqrt{y_1}} \frac{1}{1 - \beta(r_1)\frac{1}{2\sqrt{y_1}}}$$

and, for example, on the wave

$$r_1 = a_0 t, \ \sqrt{y_1} = \beta(r_1),$$

$$\frac{\partial P}{\partial t} = P_1^0 f(\theta) \frac{1}{(n+1)\frac{P_1^0}{Bn} f(\theta)\, t}$$

or

$$\frac{\partial P}{\partial t} = \frac{Bn}{t} \frac{1}{n+1}.$$

On the shock wave BB', according to (3.5),

$$\frac{\partial P}{\partial t} = \frac{Bn}{t} \frac{2}{n+1}.$$

When rarefaction arises /3/ the acceleration will have the opposite sign. On BC (Figure 5), where $y_1 = 0$,

$$\frac{\partial P}{\partial t} = -\frac{Bn}{t} \frac{1}{n+1}, \quad \frac{\partial P}{\partial r_1} = \frac{Bn}{a_0 t} \frac{1}{n+1}, \tag{4.3}$$

which follows from (3.31) and (3.18). Thus, if the case $P_a = 1$ is taken, $\frac{\partial P}{\partial r_1}$, which vanishes in front of BC, suffers a discontinuity behind BC (according to (4.3)).

Hence, near the sonic line there is a region of finite gradients, called short waves /9/. Near the point B the short wave is a function of two variables. Consider the motion in such a region. On the basis of the orders analyzed above we introduce the variables

$$r_1 = a_0 t\left(1 + \frac{n+1}{2} M_0 \delta\right),\ v_r = M_0 a_0 \mu,\ \theta - \theta_0 = \sqrt{\frac{n+1}{2} M_0}\ Y,$$

$$v_\theta = M_0^{\frac{3}{2}} a_0 \nu \sqrt{\frac{n+1}{2}},\ M_0 = \frac{P_1^0}{Bn},\ \frac{P}{Bn} = M_0 \alpha, \tag{4.4}$$

where δ, μ, Y, ν and α are $0(1)$. If leading-order small quantities are retained, the equations of motion and continuity become

$$\alpha = \mu,\ \frac{\partial \nu}{\partial \delta} - \frac{\partial \mu}{\partial y} = 0,$$

$$\frac{\partial \mu}{\partial \delta}(\mu - \delta) + \frac{1}{2}\mu + \frac{1}{2}\frac{\partial \nu}{\partial y} = 0. \tag{4.5}$$

Particular solutions of (4.5), depending on the three constants A, B, and C are sought in the forms

$$\delta = -\frac{1}{2}\frac{d\varphi}{d\mu} Y^2 + F(\mu),$$

$$\nu = \varphi(\mu)\ Y, \tag{4.6}$$

where

$$\varphi(\mu) = \frac{1}{C}\operatorname{tg}(C\mu + A) - \mu,$$

$$F(\mu) = B\sin^2(C\mu + A) + \frac{1}{2C}\sin 2(C\mu + A) + \mu.$$

Allowance for viscous terms in the vicinity of B does not yield any improvements, since for fluids the coefficient of kinematic viscosity $\nu \approx \frac{0.01}{100^2}\,\mathrm{m}^2/\mathrm{sec}$ and the order of magnitude of the viscous forces $\frac{\nu_1/a_0^2 t}{(P_1/Bn)^2} \sim \frac{10^{-12}}{t\,(P_1/Bn)^2} \ll 1$. The parameters of the point B are determined by the solution in the hyperbolic region. In the vicinity of B there is no difference to first order between the cases of arbitrary $P_a(\xi)$ and $P_a = 1$; therefore a constant value $P = P_1^0$ can be used. In the constant-flow region ABC of § 1 (Figure 5)

$$P = P_1^0,\ v = a_0 \frac{P_1^0}{Bn},\quad D = a_0\left(1 + \frac{n+1}{4}\frac{P_1^0}{Bn}\right), \tag{4.7}$$

to first order. The equation of the parabolic line for the system of equations (F) is

$$r = a + v_r - \frac{v_\theta^2}{r},$$

which becomes

$$\frac{r_1}{a_0 t} = 1 + \frac{n+1}{2}\frac{P_1^0}{Bn} - \frac{(\theta - \beta)^2}{4}\frac{P_1^0}{Bn} - \frac{3n^2 + 6n - 1}{8}\left(\frac{P_1^0}{Bn}\right)^2 \tag{4.8}$$

near the point B (Figure 5); β is the angle between the velocity and the $0x$ axis, where (according to § 2)

$$\beta = \frac{\pi}{2} - \alpha_1 - \frac{n+1}{4}\frac{P_1^0}{Bn}\operatorname{tg}\alpha_1. \tag{4.9}$$

If the equation of the parabolic line to first order, namely

$$\frac{r_1}{a_0 t} = 1 + \frac{n+1}{2}\frac{P_1^0}{Bn}, \tag{4.10}$$

is substituted in $\frac{r_1}{t} = r = \sqrt{\xi^2 + \eta^2}$, which is the equation of the shock line AB (see (2.5)), the coordinate of their point of intersection is derived in the form

$$\xi_B - a_0 \sin\alpha_1 = a_0 \sin\theta_0 \sqrt{\frac{n+1}{2}\frac{P_1^0}{Bn}}, \quad \sin\theta_0 = \cos\alpha_1$$

or

$$\theta_B = \theta_0 - \sqrt{\frac{n+1}{2}\frac{P_1^0}{Bn}}. \tag{4.11}$$

At the point B the following relationships may be used to determine the constants A, B and C:

$$P = P_1^0, \quad v_\theta = v_1(\beta - \theta), \quad v_1 = a_0\frac{P_1^0}{Bn};$$

since

$$\theta_B - \beta_B = -\sqrt{\frac{n+1}{2}\frac{P_1^0}{Bn}},$$

we have

$$v_\theta = a_0\sqrt{\frac{n+1}{2}}\left(\frac{P_1^0}{Bn}\right)^{\frac{3}{2}}.$$

Hence, from (4.4)

$$\nu_B = 1, \quad \mu_B = 1, \quad \delta_B = 1, \quad Y_B = -1. \tag{4.12}$$

The boundary condition on the segment BC, given by the constant solution in the region ABC, in this case satisfies (4.4) near B.

The velocity D given in equation (2.2) becomes

$$\frac{d\delta}{dy} = -\sqrt{2\delta - \mu}, \tag{4.13}$$

in terms of the variables (4.4).

The pressure on AB is constant in the first approximation up to the point with coordinates $\theta = \theta_B$, after which comes a region of fast decrease in P.

From formula (2.2) for the shock wave velocity

$$D = \eta\cos\alpha + \xi\sin\alpha,$$

where $\pi - \alpha$ is the angle between the tangent to the shock wave and $0x$, we have

$$\frac{\partial D}{\partial\xi} = \frac{\xi - a_0\left(1 + \frac{n+1}{4}\frac{P}{Bn}\right)}{\cos\alpha}\frac{\partial\alpha}{\partial\xi},$$

the differentiation being performed along the shock wave governed by

$$\eta = \frac{D}{\cos\alpha} - \xi \operatorname{tg}\alpha, \quad D = a_0\left(1 + \frac{n+1}{4}\frac{P}{Bn}\right).$$

The pressure is given by $\frac{\partial P}{\partial \xi} = \left(a_0\frac{n+1}{4}\right)^{-1} Bn\frac{\partial D}{\partial \xi}$. Hence it follows that near the point $\xi_B = a_0\sin\alpha_1 + a_0\cos\alpha_1\sqrt{\frac{n+1}{2}\frac{P_1^0}{Bn}}$

$$\frac{\partial P}{\partial \theta} = \frac{4}{n+1} Bn\frac{\xi - a_0\sin\alpha_1}{\cos\alpha_1} k_B,$$

where k_B is the curvature of the shock wave at B. Alternatively, in the notation of (4.4) $\frac{d\mu}{dY} = 2k_B a_0$.

Relations (4.6) are now used to determine the solution in the vicinity of the point B. According to (1.10) the linear solution in the variables δ, μ, Y near the point B is

$$\mu = \frac{1}{\pi}\operatorname{arc\,tg}\frac{\sqrt{2}\cdot\sqrt{-\delta}}{Y^2}$$

or

$$2\delta = -Y^2\operatorname{tg}^2\mu\pi. \tag{4.14}$$

If this expression is compared with (4.6), which for large δ should take the form (4.14), it is clear that $A=0, C=\pi$. This yields

$$\delta = -\frac{1}{2}Y^2\operatorname{tg}^2\mu\pi + \mu + \frac{1}{2\pi}\sin 2\mu\pi + B\cdot\sin^2\mu\pi,$$

$$\nu = \left(\frac{1}{\pi}\operatorname{tg}\pi\mu - \mu\right)Y. \tag{4.6'}$$

As one moves further away from B toward the $0y$ axis, this solution should transform to (3.8). It can be easily shown that for finite $\theta - \theta_0$ or large Y the solution for μ in § 3 is of the order of $\frac{P_1}{Bn}$. From the foregoing this gives

$$\delta = -\frac{1}{2}Y^2\pi^2\mu^2 + 2\mu, \quad \nu \sim \mu^3 Y,$$

which agrees with the solution of § 3 when approaching B. At the point B, where $Y=-1$, $\mu=1$,

$$\frac{\partial\delta}{\partial\mu} = 2, \quad \frac{\partial\delta}{\partial Y} = 0, \text{ and from (4.13)} \quad \frac{d\mu}{dY} = -\frac{1}{2}.$$

Incidentally, $\delta(\mu, y)$ from (4.14) and $\nu(\mu, y)$ from (4.6') are solutions of the linear version of system (4.5).

The shock wave equation is found by substituting (4.6') in (4.13) and integrating subject to the boundary condition $Y=-1$, $\mu=1$. The velocity component tangential to the shock vanishes to first order. We note that the solution was related to the linear solution of § 3 and the constants accordingly determined, thereby obtaining a solution uniformly valid to first order in accordance with the ideas about using a slowly varying amplitude /45/.

The constant B is chosen to fit smoothly with (3.6); when $B=0$, $\frac{d\mu}{dY}$ is nonmonotonic.

The variation of P/P_1^0 with θ along the shock wave $ABB'A'$ is given in Figure 6. The linear characteristics are now replaced by the improved ones in the vicinity of the point B on the shock wave.

In the linear formulation the solution of the two-dimensional problem for $P_a=1$ is

$$\frac{P}{P_1^0}=\frac{1}{\pi}\operatorname{arc\,tg}\frac{\sqrt{t-\frac{r_1}{a_0}}\sqrt{\frac{2a_1}{r_0}}}{\theta-\theta_0}, \qquad (4.14')$$

where first-order small terms are retained near B.

The linear characteristic variable $t-\frac{r_1}{a_0}$ is replaced by y_1, where $y_1=\text{const}$ is the equation of the improved characteristics. When approaching the point B from the region of constant flow $P=P_1^0$ and, since $\theta_B-\theta_0<0$, (4.14') indicates that $y_1=0$ (the branch of arctg x is chosen for which arctg $x=\pi$ at $x=-0$).

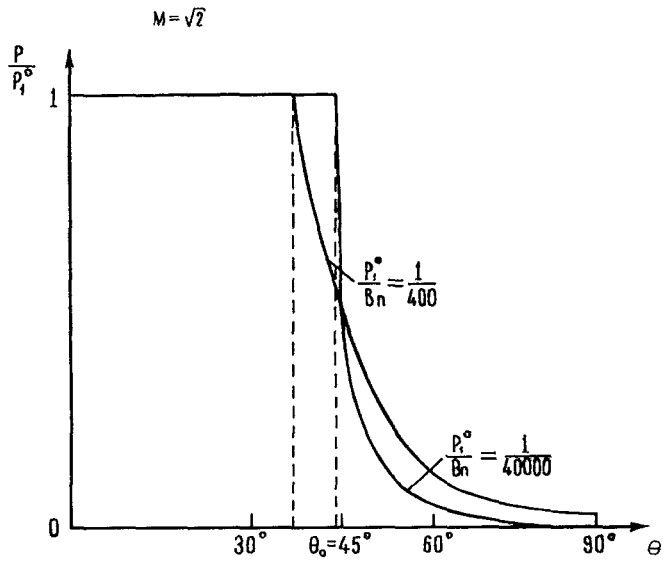


FIGURE 6.

Since $y_1=0$ in the vicinity of B, the right-hand side of (4.14') can be written in the form

$$1-\frac{1}{\pi}\frac{\sqrt{y_1}\sqrt{\frac{2a_0}{r_1}}}{\theta-\theta_0}$$

and equation (3.2) again becomes one-dimensional in r_1 near $y_1=0$. With the aid of (3.30) it is easy to derive that (4.3) holds for small y_1 in the

neighborhood of the point B. From (4.3) one finds that $\frac{\partial\mu}{\partial\delta} = \frac{1}{2}$ along the shock wave at the point B. Now,

$$\left(\frac{d\mu}{dY}\right)_B = \frac{\partial\mu}{\partial Y}\frac{\partial\delta}{\partial Y} + \frac{\partial\mu}{\partial Y} \quad \text{where} \quad \left(\frac{\partial\delta}{\partial Y}\right)_B = -1 \text{ and}$$

$$\frac{\partial\mu}{\partial Y} = \frac{1}{P_1^0}\sqrt{\frac{n+1}{2}\frac{P_1^0}{Bn}}\frac{\partial P}{\partial\vartheta},$$

and from (3.30) at the point B, $\frac{\partial P}{\partial\vartheta} = Bn\left(C' - \frac{1}{n+1}C_3\right)$, $\frac{\partial\mu}{\partial Y} = O\left(\sqrt{\frac{n+1}{2}\frac{P_1^0}{Bn}}\right)$; hence, one can derive the following results: $\frac{\partial\mu}{\partial Y} = 0$, $\left(\frac{d\mu}{dY}\right)_B = -\frac{1}{2}$, which agree with those obtained earlier (Figure 6). Thus, if the characteristic variable is replaced in the linear solution the slope of the pressure curve along the shock at B is obtained exactly to first order. It is interesting to note that the solution along BB' is correct to variable order α in P_1^0/Bn $(1 < \alpha \leqslant 2)$, a fact which is also true when moving from the shock wave BB' to the interior of the region (see § 3). The vicinity of the point B was also investigated by Bulakh /21/ and Legras /8/ for a similar problem in wing theory, but in their treatment of the transition from the region ABC of constant flow to the region BB' on the shock the pressure and velocity rise discontinuously at the point B, whereas from physical considerations it is clear that nowhere in the region $CBB'C'$ (Figure 5) does the pressure exceed the value P_1^0 in the region ABC. There is rarefaction behind the point B. Since in /21/ no solutions are obtained in the vicinity of B we shall consider Legras' work /8/. In the problem of the flow around the edge of a tetragonal wing the situation is as follows (see Figure 18 in the Appendix): above the wing $(0 < \theta < \pi)$ there is a rarefied flow and below the wing $(\pi < \theta < 2\pi)$ a compressive flow. Legras studied the vicinity of B, where the discontinuities associated with the plane (AB) and conical (MN) sections of the wing base meet. He obtained the result that at the point B the velocity jumps from $v_y = w_0$ (the normal component of the perturbed velocity at the wing) to $\frac{8}{5}w_0$, whereas in the transition to the conical section the jump produces rarefaction $(v_y < w_0)$. Mathematically, the error in the result of /8/ can be explained as follows. Legras looks for the velocity field in the parametric form

$$u = -\frac{w_0}{\beta\pi}\left(\frac{\pi}{2} - \alpha\right) + a_2 + tl_1(\alpha),$$

$$v = b_2 + tl_2(\alpha),$$

$$w = \frac{w_0}{\pi}\left(\frac{\pi}{2} - \alpha\right) + C_2 + tl_3(\alpha), \quad v_y = w,$$

$$r = 1 + l(\alpha) + t,$$

$$\theta = \frac{3\pi}{2} + m(\alpha),$$

where $l(\alpha)$, $l_1(\alpha)$, $l_3(\alpha)$ are first-order small quantities, $l_2(\alpha)$, $m(\alpha)$ are of order $\frac{1}{2}$, $b_2(\alpha)$ is of order $\frac{3}{2}$, and a_2, C_2 are of second order.

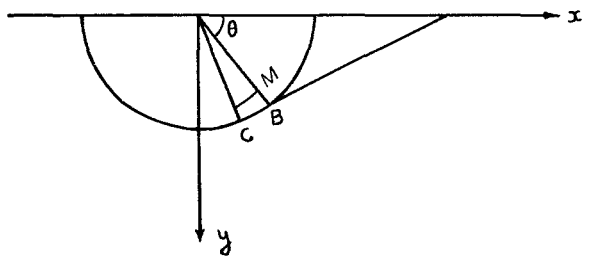


FIGURE 7.

Hence the derivatives are given by $\frac{\partial u}{\partial r} = l_1$, $\frac{\partial w}{\partial r} = l_3$, i.e., the derivatives with respect to r are first-order small quantities, as are the functions themselves. On the other hand, by the methods described at the beginning of this section it can be shown that the derivatives with respect to r are not small (short wave); the solution for tetragonal and triangular wings is given in Appendix II.

Consider now the solution of the problem in the elliptic region, for which $P = P_1^0 P_a(\xi)$, $\xi = \frac{x}{t}$ at the fluid boundary.

To determine the second approximation in the region $BCB'C'$ one must solve the equations in the second approximation. Equation (F') for the velocity potential $\varphi(x, y, t)$ is

$$a^2\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) - \left(\frac{\partial\varphi}{\partial x}\right)^2\frac{\partial^2\varphi}{\partial x^2} - \left(\frac{\partial\varphi}{\partial y}\right)^2\frac{\partial^2\varphi}{\partial y^2} -$$
$$-2\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y} - 2\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x\partial t} - 2\frac{\partial^2\varphi}{\partial y\partial t}\frac{\partial\varphi}{\partial y} - \frac{\partial^2\varphi}{\partial t^2} = 0,$$

where a, the velocity of sound in the disturbed fluid, is given by

$$a^2 = a_0^2 - (n-1)\left(\frac{1}{2}\varphi_x^2 + \frac{1}{2}\varphi_y^2 + \frac{\partial\varphi}{\partial t}\right), \quad \varphi_x = \frac{\partial\varphi}{\partial x}, \quad \varphi_y = \frac{\partial\varphi}{\partial y}.$$

If the solution is sought as a power series in the small parameter $\frac{P_1^0}{Bn}$, where

$$\varphi = \varphi_1 + \varphi_2,$$

we find that φ_1 satisfies a wave equation and the second-order quantity φ_2 is governed by the equation

$$\frac{\partial^2\varphi_2}{\partial x^2} + \frac{\partial^2\varphi_2}{\partial y^2} - \frac{1}{a_0^2}\frac{\partial^2\varphi}{\partial t^2} = \frac{1}{a_0^2}\left(\frac{n-1}{a_0^2}\frac{\partial\varphi_1}{\partial t}\frac{\partial^2\varphi_1}{\partial t^2} + 2\frac{\partial\varphi_1}{\partial x}\frac{\partial^2\varphi_1}{\partial x^2} + 2\frac{\partial\varphi_1}{\partial y}\frac{\partial^2\varphi_1}{\partial y^2}\right).$$

Differentiating this latter equation with respect to t and putting $\frac{\partial\varphi_2}{\partial t} = \Phi$, the following may be written:

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} - \frac{1}{a_0^2}\frac{\partial^2\Phi}{\partial t^2} = \psi(x, y, t),$$
$$\psi(x, y, t) = \frac{\partial}{\partial t}\left\{\frac{1}{a_0^2}\left(\frac{n-1}{a_0^2}\frac{\partial\varphi_1}{\partial t}\frac{\partial^2\varphi_1}{\partial t^2} + 2\frac{\partial\varphi_1}{\partial x}\frac{\partial^2\varphi_1}{\partial x^2} + 2\frac{\partial\varphi_1}{\partial y}\frac{\partial^2\varphi_1}{\partial y^2}\right)\right\}. \qquad (4.15)$$

Consider the case $V < a_0$. The boundary-value problem for this equation can be solved in the region $\sqrt{x^2+y^2} < a_0 t$ subject to the boundary conditions (3.9) of § 3 taken on $\sqrt{x^2+y^2} = r_1 = a_0 t$, where the singularity on the right-hand side of (4.15) near $\sqrt{x^2+y^2} = a_0 t$ is of the order of $\frac{1}{\sqrt{a_0^2 t^2 - x^2 - y^2}}$ or $\frac{1}{\sqrt{y_1}}$. In virtue of (3.6) this is of the order of $\frac{1}{P_1^0 Bn}$, whereas the region of integration is of the order of $\left(\frac{P_1^0}{Bn}\right)^2$. For the problems being studied in this section equation (4.15) expressed in Chaplygin's variables is

$$\frac{\partial^2 \Phi}{\partial \varepsilon^2} + \frac{1}{\varepsilon}\frac{\partial \Phi}{\partial \varepsilon} + \frac{1}{\varepsilon^2}\frac{\partial^2 \Phi}{\partial \sigma^2} = F_2(\varepsilon, \sigma)\frac{4a_0}{(1+\varepsilon^2)^2},$$
$$F_2 = t^{-2}\psi. \tag{4.16}$$

The boundary condition on $\sqrt{x^2+y^2} = a_0 t$ is given by (3.6) where $\Phi = -\frac{P}{\rho_0}$. The condition on the fluid surface is

$$\tilde{\varphi} - \xi\frac{\partial \tilde{\varphi}}{\partial \xi} - \eta\frac{\partial \tilde{\varphi}}{\partial \eta} + \frac{1}{2}\left(\frac{\partial \tilde{\varphi}}{\partial \xi}\right)^2 + \frac{1}{2}\left(\frac{\partial \tilde{\varphi}}{\partial \eta}\right)^2 - \frac{a^2}{n-1} = \frac{a_0^2}{n-1},$$

where $\tilde{\varphi} = \frac{\varphi}{t}$. Alternatively, since

$$\frac{a^2}{n-1} - \frac{a_0^2}{n-1} = \frac{P}{\rho_0} - \frac{1}{\rho_0}\frac{P^2}{Bn},$$

we obtain on the boundary $\eta = \eta_1(\xi)$ in the first and second approximations

$$\tilde{\varphi}_1 - \xi\frac{\partial \tilde{\varphi}_1}{\partial \xi} + \frac{P_1^0 P_a(\xi)}{\rho_0} = 0, \quad \eta_1 - \xi\frac{d\eta_1}{d\xi} = \frac{\partial \tilde{\varphi}}{\partial \eta}. \tag{4.17}$$

$$\tilde{\varphi}_2 - \xi\frac{\partial \tilde{\varphi}_2}{\partial \xi} - \xi\frac{\partial^2 \tilde{\varphi}_1}{\partial \eta \partial \xi}\eta_1 + \frac{1}{2}\left(\frac{\partial \tilde{\varphi}_1}{\partial \xi}\right)^2 + \frac{1}{2}\left(\frac{\partial \tilde{\varphi}_1}{\partial \eta}\right)^2 + \frac{1}{\rho_0}\frac{\partial P_1}{\partial \eta}\eta_1 - \frac{1}{\rho_0}\frac{P_1^2}{2Bn} + \frac{P_2}{\rho_0} = 0, \tag{4.18}$$

where $P = P_1 + P_2$; P_2 vanishes on the boundary, $P_1 = P_1^0 P_a(\xi)$, and the conditions (4.17), (4.18) are satisfied for $\eta = 0$, i.e., on the undisturbed boundary. It is therefore necessary to solve equation (4.16) subject to boundary conditions (3.6) and (4.18).

It can be easily shown that as $\eta \to 0$, the singularity at the points $\xi = 0$, $\xi = V$ is identical to that for an incompressible fluid /27/.

The solution for an incompressible fluid has the singularity

$$\frac{\partial \varphi}{\partial \eta} \sim \ln\frac{V-\xi}{\xi}, \quad \eta_1 \sim \ln\xi + C(\xi), \tag{4.19}$$

where $C(\xi)$ is continuous for $\xi = V$ and $\xi = 0$.

Hence, when the equation

$$\frac{1}{\rho_0}\frac{\partial P_1}{\partial \eta_1} = \xi\frac{\partial^2 \tilde{\varphi}_1}{\partial \xi \partial \eta_1} - \frac{\partial \tilde{\varphi}_1}{\partial \eta} \tag{4.20}$$

is used in (4.18) the third and sixth terms cancel and

$$\frac{\partial \tilde{\varphi}_2}{\partial \xi} \sim \ln^2\frac{V-\xi}{\xi}. \tag{4.21}$$

The right-hand side of (4.15) can thus be expressed as

$$\frac{1}{a_0^2}\frac{\partial^2}{\partial t^2}\left\{\left(\frac{\partial\varphi_1}{a_0\partial t}\right)^2(n-1)+2\left(\frac{\partial\varphi_1}{\partial x}\right)^2+2\left(\frac{\partial\varphi_1}{\partial y}\right)^2\right\}, \tag{4.22}$$

where the expression in the brackets in the case $P_a=1$ is given by (1.11), (1.12).

Further, for a subsonic velocity $V<a_0$ one can obtain a solution by the procedure developed for conical flows in /28/. It will not be as easy to solve the more interesting case of arbitrary $P_a(\xi)$, since one cannot obtain such simple relations as (1.11) and (1.12) for v_x and v_y.

For supersonic motion over the surface ($V>a_0$) the motion is described by Figure 5; the pressures on the arc BB' and the corresponding arc $r_1=a_0t$ (B_1B_1') are given by (3.6) and (3.8) respectively. A second-order displacement along r_1 changes the boundary condition in the second approximation. Near the point B the boundary condition in the second approximation is unobtainable, since the shock-wave type of solution (4.4) holds in the first approximation. However, a region of the order of $\theta-\theta_0\sim\left(\frac{P_1^0}{Bn}\right)^{\frac{1}{2}}$ cannot affect the solution in the second approximation, since the integral over this region with respect to P_2 (which is second order) will be of higher order than second.

However, when solving the problem for $V>a_0$ difficulties arise when formulating the boundary conditions on the arc B_1C_1 (Figure 5).

The conditions are known on the arc BC of the parabolic line

$$r=a+v_r-\frac{V_0^2}{2r} \tag{4.23}$$

or, more accurately, on the envelope of the characteristics (see § 2).

The conditions cannot be taken on the line $r=a_0$, since when passing to $r=a_0$ (the sonic line of the linear problem) the solution changes in the first, and all the more so in the second order.

Thus, Chaplygin's transformation does not aid here in simplifying the problem, since for a Laplace equation with the right-hand side as in (4.16) the boundary conditions are not given on the arc B_1C_1 (Figure 5).

The attempt in /21/ to overcome this difficulty was unsuccessful, since the technique used there to solve the problem in the boundary layer is only valid for second order displacements along r; the boundary conditions in /21/ are satisfied on the curve $\hat{r}=\hat{r}_1$, which represents the Mach cone of the exact problem, and there is no guarantee that the conditions on the Mach cone $r=r_1$ of the linear problem are satisfied. In this connection, there is a peculiarity in the solution by the indicated method; for example, in formula (3.4.24) of /21/ use is made of the following expansion in δ which is invalid for a first-order displacement: $(r_1-r+\delta r)^{3/2}=(r_1-r)^{3/2}+\frac{3}{2}\frac{\delta r}{(r_1-r)^{1/2}}$. This difficulty can be overcome by using the result of § 3 and introducing the expansion for r in the form

$$r=R+\gamma r_1(\theta)+\gamma^2 r_2(\theta), \tag{4.24}$$

where $\gamma=\frac{P_1^0}{Bn}$, $r_1(\theta)=0$ for a part of the boundary BB' of the elliptic region,

and $\gamma r_1(\theta) = v_r + \frac{n-1}{2} a_0 \frac{P_1^0}{Bn}$ on BC (according to § 3). The equation for the velocity potential $\tilde{\varphi} = t\bar{\varphi}$ in the variables r, θ is

$$\frac{\partial^2\bar{\varphi}}{\partial r^2}\left(\left(\frac{\partial\bar{\varphi}}{\partial r}-r\right)^2-a^2\right)+\frac{1}{r^2}\left(2r-\frac{\partial\bar{\varphi}}{\partial r}\right)\left(\frac{\partial\bar{\varphi}}{\partial\theta}\right)^2-2\frac{1}{r^2}\frac{\partial\bar{\varphi}}{\partial\theta}\left(r-\frac{\partial\bar{\varphi}}{\partial r}\right)\frac{\partial^2\bar{\varphi}}{\partial r\partial\theta}+$$

$$+\frac{1}{r^4}\left(\left(\frac{\partial\bar{\varphi}}{\partial\theta}\right)^2-r^2a^2\right)\frac{\partial^2\bar{\varphi}}{\partial\theta^2}-\frac{a^2}{r}\frac{\partial\bar{\varphi}}{\partial r}=0, \qquad (4.25)$$

where

$$a^2 = a_0^2 - \frac{n-1}{2}\left(\frac{\partial\bar{\varphi}}{\partial r}\right)^2-\frac{n-1}{2}\frac{1}{r^2}\left(\frac{\partial\bar{\varphi}}{\partial\theta}\right)^2-(n-1)\left(\bar{\varphi}-r\frac{\partial\bar{\varphi}}{\partial r}\right).$$

If the variables r, θ are transformed to R, θ (according to (4.24)) in (4.25), where $\bar{\varphi} = \gamma\bar{\varphi}_0(R,\theta) + \gamma^2\bar{\varphi}_1(R,\theta)$ and to second order

$$\frac{\partial\bar{\varphi}}{\partial R} = \frac{\partial\bar{\varphi}}{\partial r}, \frac{\partial^2\bar{\varphi}}{\partial R^2} = \frac{\partial^2\bar{\varphi}}{\partial r^2}, \left.\frac{\partial\bar{\varphi}}{\partial\theta}\right|_R = \left.\frac{\partial\bar{\varphi}}{\partial\theta}\right|_r + \frac{\partial\varphi}{\partial R}\gamma\frac{\partial r_1}{\partial\theta},$$

$$\left.\frac{\partial^2\bar{\varphi}}{\partial\theta^2}\right|_R = \left.\frac{\partial^2\bar{\varphi}}{\partial r^2}\right|_r + 2\frac{\partial^2\bar{\varphi}}{\partial R\partial\theta}\gamma\frac{\partial r_1}{\partial\theta}+\frac{\partial\bar{\varphi}}{\partial\theta}\gamma\frac{\partial^2 r_1}{\partial\theta^2},$$

and then terms to the first power of γ are equated, we obtain the usual linearized equation for $\bar{\varphi}_0$ in the variables R, θ. When terms to the second power of γ are equated, we derive the equation

$$\frac{\partial^2\bar{\varphi}_1}{\partial R^2}(R^2-a_0^2)-\frac{a_0^2}{R^2}\frac{\partial^2\varphi_1}{\partial\theta^2}-\frac{a_0^2}{R}\frac{\partial\bar{\varphi}_1}{\partial R} = -\frac{\partial^2\bar{\varphi}_0}{\partial R^2}(2Rr_1-(n+1)\frac{\partial\bar{\varphi}_0}{\partial R}R+$$

$$+(n-1)\bar{\varphi}_0)-2\frac{1}{R^2}\left(\frac{\partial\bar{\varphi}_0}{\partial\theta}\right)^2+\frac{2}{R}\frac{\partial\bar{\varphi}_0}{\partial\theta}\frac{\partial^2\bar{\varphi}_0}{\partial\theta\partial R}+\frac{a_0^2}{R^2}\left(-2\frac{\partial^2\bar{\varphi}_0}{\partial R\partial\theta}\frac{\partial r_1}{\partial\theta}-\frac{\partial^2 r_1}{\partial\theta^2}\frac{\partial\bar{\varphi}_0}{\partial R}\right)-$$

$$-\frac{1}{R^2}\left((n-1)\left(\bar{\varphi}_0-R\frac{\partial\bar{\varphi}_0}{\partial R}\right)-\frac{a_0^2}{R}2r_1\right)\frac{\partial^2\bar{\varphi}_0}{\partial\theta^2}-\frac{1}{R}\left((n-1)\left(\bar{\varphi}_0-R\frac{\partial\bar{\varphi}_0}{\partial R}\right)+\frac{a_0^2}{R}r_1\right)\frac{\partial\bar{\varphi}_0}{\partial R}. \qquad (4.26)$$

The only term containing a singularity at $R = a_0$ will be the first one on the right-hand side of (4.26), but if one chooses r_1 in (4.24) so that the expression in parentheses multiplying $\frac{\partial^2\bar{\varphi}_0}{\partial R^2}$ vanishes at $R = 1$ then the singularity in equation (4.26) is eliminated. Differentiating (4.26) with respect to t and using a Chaplygin transformation to obtain an equation for $\frac{\partial\varphi}{\partial t}$ in the variables R, θ (a Laplace equation with a right-hand side), the singularity on the right-hand side will be integrable (see the considerations in the derivation of (4.15)).

Near the point B on BB', r_1 is not equal to zero and can be obtained according to the theory of short waves (Figure 6). Thus, by introducing the variables R, θ after a Chaplygin transformation, the problem in the second approximation inside the diffraction region $CBB'C'$ (Figure 5) reduces to solving a Laplace equation with a known right-hand side in the semicircle $R = a_0$. The boundary condition $\theta = 0$, $\theta = \pi$ is given by (4.18); for $R = a_0$, $P = P_1^0$ on BC and P on BB' can be found from (3.8). It can be shown, however,

that the condition $P=P_1^0$ on $R=a_0$ is not exact to second order (the exact condition is $P=P_1^0$ on BC). The problem for $v>a_0$ is not solved analytically. The solution /65/ is not rigorous.

When solving the general problem in the second approximation in the region $ABB'A'$ (Figure 5), one cannot simply carry over the boundary condition for the potential φ on the line AB ($\varphi=0$) to the line AB_1, which represents the shock wave of the linear problem, and set $\varphi_2=0$ on it. In fact, from § 1 for constant flow and $P_a=1$ the potential φ is in the region ABC

$$\frac{\varphi}{t}=\frac{P_1^0}{Bn}(\xi\cos\beta+\eta\sin\beta-V\cos\beta),$$

where β, the angle between the normal to the front and the $0x$ axis, is

$$\beta=\frac{\pi}{2}-\alpha_1-\frac{n+1}{4}\frac{P_1^0}{Bn}\operatorname{tg}\alpha_1.$$

Thus, on the line AB_1 ($\xi\sin\alpha_1+\eta\cos\alpha_1-V\sin\alpha_1=0$) $\frac{\varphi}{t}-\frac{\varphi_1}{t}=\frac{\varphi_2}{t}$ does not vanish.

The solution in the second approximation can also be looked for in terms of the variables x, y, t.

Suppose the velocity potential φ has the form

$$\varphi=\gamma\varphi_1(u,x,y)+\gamma^2\varphi_2(u,x,y), \tag{4.27}$$

$$t=u+\gamma t_1(u,x,t), \tag{4.28}$$

where $\gamma=\frac{P_1^0}{\rho_0 a_0^2}$; the following equations are then found for φ_1 and φ_2:

$$\frac{\partial^2\varphi_1}{\partial u^2}-a_0^2\frac{\partial^2\varphi_1}{\partial x^2}-a_0^2\frac{\partial^2\varphi_1}{\partial y^2}=0, \tag{4.29}$$

$$\frac{\partial^2\varphi_2}{\partial u^2}-a_0^2\frac{\partial^2\varphi_2}{\partial x^2}-a_0^2\frac{\partial^2\varphi_2}{\partial y^2}=2\frac{\partial^2\varphi_1}{\partial u^2}\left(\frac{\partial t_1}{\partial u}-\frac{n-1}{2}\frac{1}{a_0^2}\frac{\partial\varphi_1}{\partial u}\right)-2\frac{\partial^2\varphi_1}{\partial x\partial u}\left(a_0^2\frac{\partial t_1}{\partial x}+\frac{\partial\varphi_1}{\partial x}\right)-$$

$$-2\frac{\partial^2\varphi_1}{\partial y\partial u}\left(a_0^2\frac{\partial t_1}{\partial y}+\frac{\partial\varphi_1}{\partial y}\right)+\frac{\partial\varphi_1}{\partial u}\left(\frac{\partial^2 t_1}{\partial u^2}-a_0^2\frac{\partial t_1}{\partial x^2}-a_0^2\frac{\partial^2 t_1}{\partial y^2}\right). \tag{4.30}$$

The right-hand side of (4.30), where φ_1 is found from (3.1), has a singularity near $r_1=a_0t$. The quantity t_1 was chosen so that the coefficients of $\frac{\partial^2\varphi_1}{\partial u^2}$, $\frac{\partial^2\varphi_1}{\partial x\partial u}$, $\frac{\partial^2\varphi_1}{a y\partial u}$ in (4.30) vanish on $r_1=a_0t$. This yields

$$t_1=-\frac{n-1}{2}t-\frac{r_1}{a_0}\cos(\beta-\theta), \tag{4.31}$$

where β is the angle between the normal to AB and the $0x$ axis, and $v_{r_1}=\frac{P_1^0}{\rho_0 a_0}\cos(\beta-\theta)$. Formulas (4.28), (4.31) and (4.24) obviously all give the same result.

To solve the problem in the second approximation in the whole region $ABB'A'$ for an arbitrary pressure on the surface in both the two-dimensional and axisymmetric cases, the required parameters, as well as the independent variables are expressed as series in a small parameter. Clarkson /41/

showed that for the steady-state flow around a profile this enables the solution to be determined in the second approximation in the whole region of the motion.

If in the equation for the potential $\varphi(x, y, t)$, namely

$$a^2\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)-\left(\frac{\partial\varphi}{\partial x}\right)^2\frac{\partial^2\varphi}{\partial x^2}-\left(\frac{\partial\varphi}{\partial y}\right)^2\frac{\partial^2\varphi}{\partial y^2}-2\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}-$$

$$-2\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x\partial t}-2\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial y\partial t}-\frac{\partial^2\varphi}{\partial t^2}=0,$$

$$\frac{a^2}{n-1}=\frac{a_0^2}{n-1}-\frac{v^2}{2}-\frac{\partial\varphi}{\partial t},\quad v^2=\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2, \tag{4.32}$$

we introduce the expansions

$$\varphi=\gamma\varphi_1(u, x, y)+\gamma^2\varphi_2(u, x, y),\quad P=P_1'+P_2, \tag{4.33}$$

$$t=u+\gamma t_1(u, x, y),\quad \gamma=\frac{P_1^0}{Bn},\quad P_1^0=P_\Phi(0), \tag{4.34}$$

the first and second approximations are found to be

$$\frac{\partial^2\varphi_1}{\partial u^2}-a_0^2\frac{\partial^2\varphi_1}{\partial x^2}-a_0^2\frac{\partial^2\varphi_1}{\partial y^2}=0, \tag{4.35}$$

$$\frac{\partial^2\varphi_2}{\partial u^2}-a_0^2\frac{\partial^2 t_2}{\partial x^2}-a_0^2\frac{\partial^2\varphi_2}{\partial y^2}=2\frac{\partial^2\varphi_1}{\partial u^2}\left(\frac{\partial t_1}{\partial u}-\frac{n-1}{2}\frac{1}{a_0^2}\frac{\partial\varphi_1}{\partial u}\right)-2\frac{\partial^2\varphi_1}{\partial x\partial u}\left(a_0^2\frac{\partial t_1}{\partial x}+\frac{\partial\varphi_1}{\partial x}\right)-$$

$$-2\frac{\partial^2\varphi_1}{\partial y\partial u}\left(a_0^2\frac{\partial t_1}{\partial y}+\frac{\partial\varphi_1}{\partial y}\right)+\frac{\partial\varphi_1}{\partial u}\left(\frac{\partial^2 t_1}{\partial u^2}-a_0^2\frac{\partial^2 t_1}{\partial x^2}-a_0^2\frac{\partial^2 t_1}{\partial y^2}\right); \tag{4.36}$$

the conditions at $y=0$, obtained from (C) and (4.32), are

$$\gamma\frac{\partial\varphi_1}{\partial u}+\frac{P_1(|x|, u)}{\rho_0}=0,\quad P_1'(x, 0, u)=P_1, \tag{4.37}$$

$$\gamma^2\frac{\partial\varphi_2}{\partial u}-\frac{1}{2}\frac{P_1^2}{\rho_0 Bn}+\frac{v^2}{2}+\gamma\frac{\partial^2\varphi_1}{\partial y\partial u}y_1+\gamma\frac{1}{\rho_0}\frac{\partial P_1}{\partial y}y_1=0, \tag{4.38}$$

where the derivatives with respect to t are replaced by derivatives with respect to u, since it may be assumed that $t_1=t_1(x, y)$. The function $y_1(x, t)$ represents the boundary in the linear case, the last two terms on the left-hand side of (4.38) disappearing in virtue of (4.37). The solution of equation (4.35) subject to the boundary condition (4.37) is the solution of the linear problem and was found in /27/ (see also Chapter II, § 2). The pressure near the front AB_1 in the linear problem can be found in powers of the distance $u-\tau_1(x, y)$ from the front, where $u=\tau_1(x, y)$ is the equation of the linear front AB_1 (/27/, p. 122). Thus, near the front in the linear problem

$$\varphi_1=\int_0^{u-\tau_1}f_1(x, y)H(u')'du-\frac{1}{2}f_2(x, y)(u-\tau_1), \tag{4.39}$$

where $\frac{1}{\rho_0}f_1$ and $\frac{1}{\rho_0}f_2$ are the values of the pressure P_1' and its derivative on

AB_1 and $H(x)$ is a unit function. We denote the right-hand side of (4.36) by $f(u, x, y)$ and the boundary value $\varphi_2(u, x, 0)$ according to (4.38) by $g(u, x)$. The solution of (4.36) subject to condition (4.38) and zero initial values is sought using Hadamard's theory of the principal part of the integral /27/.
In the space $\xi = t', \eta = x', \zeta = y'$, a characteristic conoid is drawn through the point with coordinates $t' = u$, $x' = x$, $y' = y$ /27, 41/, which intersects the (t', x')-plane in S'', and the (x', y')-plane in S' where $y' > 0$. Through the point $(u, x, -y)$ is drawn a characteristic conoid, which intersects the (t', x')-plane in S'', and the (x', y')-plane in some region S_1 where $y' > 0$. The volume bounded by the first conoid, S'' and S' is denoted by U, and U' represents the volume bounded by the second conoid, S'' and S_1. Let the form of solution at the point (u, x, y) be

$$\frac{1}{\sqrt{(u-t')^2 - \frac{(x-x')^2}{a_0^2} - \frac{(y-y')^2}{a_0^2}}}; \tag{4.40}$$

the corresponding solution at $(u, x, -y)$ is obtained by changing the sign of y. Applying Green's formula to the volumes U and U', and using the zero initial data together with the fact that the integrals over the conoid do not make a finite nonzero contribution, we obtain

$$\varphi_2(u, x, y) = \frac{1}{2\pi a_0^2} \iiint_U \frac{f(t', x', y')\, dt'dx'dy'}{\sqrt{(u-t')^2 - \frac{(x-x')^2}{a_0^2} - \frac{(y-y')^2}{a_0^2}}} -$$
$$- \frac{1}{2\pi a_0^2} \iiint_{U'} \frac{f(t', x', y')\, dt'dx'dy'}{\sqrt{(u-t')^2 - \frac{(x-x')^2}{a_0^2} - \frac{(y+y')^2}{a_0^2}}} -$$
$$- \frac{1}{\pi} \frac{\partial}{\partial y} \iint_{S'} \frac{g(t', x')\, dt'dx'}{\sqrt{(u-t')^2 - \frac{(x-x')^2}{a_0^2} - \frac{y^2}{a_0^2}}}, \tag{4.41}$$

where $f(u, x, y)$ is the right-hand side of (4.36). Suppose the equation of the shock wave in the linear case, $t = \tau_1(x, y)$, gives $AB_1B_1'A'$ (Figure 5). For $R'(t) = V$, the equation of AB_1 will be $t = \frac{y}{V}\sqrt{\frac{V^2}{a_0^2} - 1} + \frac{x}{V}$, the indicated plane in the (t', x', y') space passing through the point $(0, 0, 0)$ and being tangent to the cone $x^2 + y^2 = a_0^2 t^2$. Since the solution $\gamma\varphi_1$ of the linear problem differs from zero in the region inside the shock wave $AB_1B_1'A'$, the indicated volumes in expression (4.41) should be taken as the volumes U and U' subject to the additional condition $t' \geqslant \tau_1(x', y')$.

Let us now consider the points lying on $u = \tau_1(x, y)$ and require that the potential φ_2 vanishes on it (φ_1 obviously equals zero for $u = \tau_1$). When the vertex (u, x, y) of the conoid is on the surface $t' = \tau_1(x', y')$, the conoid lies below this surface and is tangent to it along a straight line. We show this when $R'(t) = V$ is constant. Then $\tau_1(x, y) = \frac{y\sqrt{V^2/a_0^2 - 1}}{V} + \frac{x}{V}$. The proof is similar in the general case of arbitrary $R(t)$, but the fact is itself geometrically clear. From the equation of the conoid $t' = t - \frac{1}{a_0}\sqrt{(x'-x)^2 + (y'-y)^2}$

we have

$$\frac{\partial t'}{\partial x'} = -\frac{x' - x}{a_0 \sqrt{(x'-x)^2 + (y'-y)^2}} = \frac{\cos \varphi_0}{a_0},$$

$$\frac{\partial t'}{\partial y'} = -\frac{y' - y}{a_0 \sqrt{(x'-x)^2 + (y'-y)^2}} = \frac{\sin \varphi_0}{a_0},$$

and the equation of the surface $t' = \tau_1(x', y')$ gives $\frac{\partial t'}{\partial x'} = \frac{1}{V}$, $\frac{\partial t'}{\partial y'} = \frac{\sqrt{V^2/a_0^2 - 1}}{V}$. Hence, one of the straight lines passing through the conoid vertex will lie in the plane $t' = \tau_1(x', y')$, where φ_0 is the angle between the line's projection on the (x, y)-plane and Ox', given by $\cos \varphi_0 = \frac{a_0}{V}$.

Thus, when the point (u, x, y) lies on the wave $u = \tau_1(x, y)$, the volume U degenerates into a line and the continuous parts of f in (4.41) vanish.

Suppose the solution φ_1 near the shock wave $u = \tau_1(x, y)$ has the form (cf. (4.39))

$$\varphi_1 = \int_0^{u-\tau_1} f_1(x, y)\, H(u')\, du', \tag{4.42}$$

where $H(x)$ is a unit step function and $\tau_1 = t$ is the equation of AB_1 (Figure 5).

From (4.41) on $u = \tau_1(x, y)$, nonzero values will be given by the integral over U, where the only terms retained in f are those containing the factor

$$2\frac{\partial t_1}{\partial t'}\delta(t' - \tau_1) + 2a_0^2 \frac{\partial \tau_1}{\partial x'}\frac{\partial t}{\partial x'}\delta(t' - \tau_1) + 2a_0^2 \frac{\partial \tau_1}{\partial y'}\frac{\partial t_1}{\partial y'}\delta(t' - \tau_1) -$$

$$-2\frac{n-1}{2}\frac{f_1}{a_0^2}\delta(t' - \tau_1) H(t' - \tau_1) - 2\left(\frac{\partial \tau_1}{\partial x'}\right)^2 f_1 H(t' - \tau_1)\delta(t' - \tau_1) -$$

$$-2\left(\frac{\partial \tau_1}{\partial y'}\right)^2 f_1 H(t' - \tau_1)\delta(t' - \tau_1), \tag{4.43}$$

$\delta(x)$ being the delta function.

It is also necessary that φ_2 equals zero on $u = \tau_1$. The nonzero term in (4.41) contains the integral over U of (4.43). If we take into account that $\left(\frac{\partial \tau_1}{\partial x'}\right)^2 + \left(\frac{\partial \tau_1}{\partial y'}\right)^2 = \frac{1}{a_0^2}$ and

$$\int \delta(u - \tau_1)\, du = 1, \quad \int \delta(u - \tau_1) H(u - \tau_1)\, du = \frac{1}{2},$$

and also that the integral over the (x', y')-plane (to which the integral over U reduces) is finite when the point is situated on $u = \tau_1(x, y)$, a condition is derived for the vanishing of φ_2 on AB in the form

$$a_0^2 \frac{\partial \tau_1}{\partial x}\frac{\partial t_1}{\partial x} + a_0^2 \frac{\partial \tau_1}{\partial y}\frac{\partial t_1}{\partial y} - \frac{n+1}{4}\frac{f_1}{a_0^2} = 0, \tag{4.44}$$

where $f_1 = -\frac{P_1'}{\rho_0}$ in (4.42), $\frac{\partial t_1}{\partial u} = 0$, since it can be assumed that $t_1 = t_1(x, y)$. The term with $g(u, x)$ in (4.41) for $u = \tau_1$ contains φ_2 at the point A; φ_2 is determined up to a constant, taken equal to zero. Equation (4.44) is obtained from (4.43) by integration with respect to $u = t'$.

We show now that the integral with respect to x' and y' is finite. If, in the integral over U in (4.41), the expression (4.43) is removed from the integral after the operation of the delta function the following integral with respect to x' and y' is obtained:

$$\iint \frac{dx'dy'}{\sqrt{(\tau_1(x', y')-u)^2-\frac{(x'-x)^2}{a_0^2}-\frac{(y'-y)^2}{a_0^2}}},$$

taken over the projection on the (x, y)-plane of the region in which the conoid intersects the plane $\tau_1 = \frac{y'\sqrt{\frac{V^2}{a_0^2}-1}}{V}+\frac{x'}{V}$.

Suppose $u-\frac{y\sqrt{V^2/a_0^2-1}}{V}-\frac{x}{V}=\gamma_1$ is small. This region of intersection has the following form in the polar coordinates $x'-x=-r\cos\varphi$, $y'-y=-r\sin\varphi$:

$$u-\frac{r}{a_0}=\frac{y\sqrt{\frac{V^2}{a_0^2}-1}}{V}+\frac{x}{V}-r\frac{\sqrt{\frac{V^2}{a_0^2}-1}\sin\varphi+\cos\varphi}{V}$$

or

$$\varphi-\varphi_0=\pm\sqrt{\frac{2a_0\gamma_1}{r}},$$

where

$$\cos\varphi_0=\frac{a_0}{V}.$$

Using the fact that for U we have $y'>0$, $r\sin\varphi<y$, the integral now becomes

$$2\int_0^{\frac{y}{\sin\varphi_0}} r\,dr\int_0^{\sqrt{\frac{2a_0\gamma_1}{r}}} d\varphi'\frac{1}{\sqrt{\gamma_1^2+2\gamma_1\frac{r}{a_0}-\frac{r^2}{a_0^2}\varphi'^2}},\quad \varphi'=\varphi-\varphi_0,$$

where $\frac{y}{\sin\varphi_0}$ is a finite quantity. For $\varphi_1<\varphi'<\pi$ (where φ_1 is some angle), r is small and the integral over U is likewise small.

Making use of the smallness of γ_1, the integral with respect to r and φ' yields

$2a_0\int_0^{\frac{y}{\sin\varphi_0}} arc\sin 1\,dr=\pi a_0\frac{y}{\sin\varphi_0}$, i.e., a finite quantity. Condition (4.44) is thus substantiated.

It is now shown that this condition agrees with the equation of the shock AB (Figure 5), i.e., $t=\tau_1+\gamma t_1$. The formula for the shock wave velocity,

namely

$$\sqrt{\left(\frac{\partial t}{\partial x}\right)^2+\left(\frac{\partial t}{\partial y}\right)^2}=\frac{1}{a_0}-\frac{1}{a_0}\frac{n+1}{4}\frac{P}{Bn}$$

yields $\gamma\frac{\partial\tau_1}{\partial x}\frac{\partial t_1}{\partial x}+\gamma\frac{\partial\tau_1}{\partial y}\frac{\partial t_1}{\partial y}=-\frac{n+1}{4a_0^2}\frac{P}{Bn}$, which agrees with (4.36). Hence it follows that one should replace γt_1 in (4.34) by the adjustment to the linear shock equation (see for example (2.5), (5.12), as well as (4.17), Chapter II). Then, in terms of the variables u, x, y, the problem reduces to the quadratures (4.41), i.e., to solving a wave equation with a right-hand side in the half-space (u, x, y). For arbitrary $R(t)$, the double-integration region is bounded by the line of intersection of the surface $t'=t-\frac{\sqrt{(x'-x)^2+(y'-y)^2}}{a_0}$ and the wave front $t'=\tau_1(x', y')$ (see Chapter II, § 4), where $x'=R(t_0)+$ $+a_0(t'-t_0)\sin\alpha_1(t_0)$, $y'=a_0(t'-t_0)\cos\alpha_1(t_0)$. The point (x, y) is determined from the equation of the front $\tau_1(x, y)=\tau_1^*$, where

$$x=R(\bar t_0)+a_0(\tau_1-\bar t_0)\sin\alpha_1(\bar t_0),\quad y=a_0(\tau_0-\bar t_0)\cos\alpha_1(\bar t_0).$$

If we write $t_0=\bar t_0+\tilde t_0$, where $\tilde t_0$ is small, and retain small quantities of order $\tilde t_0^2$, then in terms of the variables defined by $x'=x-r\sin\alpha, y'=y-r\cos\alpha$ the integration region is

$$r(\alpha-\alpha_1)=\pm\sqrt{\frac{2a_0\gamma_1 r}{1-\frac{r\sin^2\alpha_1 R''}{A}}},\qquad(4.45)$$

$$A=\frac{a_0\cos^2\alpha_1}{\sin\alpha_1}-a_0R''\frac{t-\bar t_0-r/a_0}{R'^2},\quad \gamma_1=t-\tau_1(x, y).$$

To obtain the radicand in the integral with respect to x' and y', the above relations should be used, where t' is no longer equal to $t-\frac{r}{a_0}$. Retaining small orders of γ_1, the relation

$$\tilde t_0\frac{1}{\cos\alpha_1}=-\sqrt{\frac{r^2-a_0^2(\tau_1-t')^2}{A^2-A\frac{r}{a_0}\sin^2\alpha_1 R''}}$$

is derived together with the following equation for t':

$$a_0(\tau_1-t')\sin\alpha_1-r\sin\alpha=-\sqrt{\frac{r^2-a_0^2(\tau_1-t')^2}{1-\frac{r}{Aa_0}\sin^2\alpha_1 R''}}\cos\alpha_1.$$

Finally, for the radicand in the integral with respect to x' and y' we obtain

$$(t'-t)^2-\frac{r^2}{a_0^2}=2\gamma_1\frac{r}{a_0}-\frac{r^2}{a_0^2}\left(1-\frac{r}{Aa_0}\sin^2\alpha_1 R''\right)(\alpha-\alpha_1)^2;$$

when passing to the front AB_1, as in the case $R'(t)=\text{const}$, the integral is clearly

$$\pi a_0\int_0^{\frac{y}{\cos\alpha_1}}\left(1-\frac{r}{Aa_0}\sin^2\alpha_1 R''\right)^{-\frac{1}{2}}dr.$$

* Since the point (u, x, y) approaches the front along a vertical, x and y always satisfy the equation of the front.

It is thus shown that (4.41) can be used to determine the solution to second order, where u is related to t through (4.34) and γt_1 is a nonlinear adjustment to the linear equation of the shock wave. The given investigation of the integration region in U near the shock wave $u = \tau_1(x, y)$ helps in obtaining the pressure on the shock to second order. To investigate the singular integrals in (4.41) near the wave front, it is convenient to introduce the generalized function /23/

$$D = x^{\lambda+1} \text{ for } x > 0,\ D = 0 \text{ for } x < 0, \tag{4.46}$$

which is obtained from the function $D(\alpha, \lambda, x)$ /41/, by replacing α by λ and substituting $\lambda = 0$ in $D(\alpha, \lambda, x)$. Consider first $R'(t) = \text{const}$; (4.39) then becomes

$$\varphi_1 = f_1(x, y)\, D - f_2 \int_0^{u-\tau_1} D\, du \tag{4.47}$$

in the limit $\lambda \to 0$, and the right-hand side of (4.36) becomes

$$\begin{aligned}
\frac{1}{2} f = f_1(\lambda+1)\,\lambda\zeta^{\lambda-1}\Big(&- \frac{n+1}{2}\,\frac{\lambda+1}{a_0^2} f_1 \zeta^{\lambda} + a_0^2 \frac{\partial\tau_1}{\partial x'}\frac{\partial t_1}{\partial x'} + \\
&+ a_0^2 \frac{\partial\tau_1}{\partial y'}\frac{\partial t_1}{\partial y'}\Big) - f_2(\lambda+1)\,\zeta^{\lambda}\Big(-\frac{n+1}{2}\,\frac{\lambda+1}{a_0^2} f_1\zeta^{\lambda} + a_0^2\frac{\partial\tau_1}{\partial x'}\frac{\partial t_1}{\partial x_1} + \\
&+ a_0^2 \frac{\partial\tau_1}{\partial y'}\frac{\partial t_1}{\partial y'}\Big) + \frac{\partial f_1}{\partial x'}(\lambda+1)\,\zeta^{\lambda}\Big(a_0^2\frac{\partial t_1}{\partial x'} - f_1\frac{\partial\tau_1}{\partial x'}\zeta^{\lambda}\Big) + \frac{\partial f_1}{\partial y'}(\lambda + \\
&+1)\,\zeta^{\lambda}\Big(a_0^2\frac{\partial t_1}{\partial y'} - f_1\frac{\partial\tau_1}{\partial y'}\zeta^{\lambda}\Big) + f_1(\lambda+1)\,\zeta^{\lambda}\Big(-a_0^2\frac{\partial^2 t_1}{\partial x'^2} - a_0^2\frac{\partial^2 t_1}{\partial y'^2}\Big),
\end{aligned} \tag{4.48}$$

where

$$\zeta = t' - \tau_1(x', y') \tag{4.49}$$

(it is assumed that $t_1 = t_1(x', y')$). The first term in (4.48) gives the following integral for the expression for φ_2 in (4.41):

$$\frac{1}{2\pi a_0^2}\, 2 \int_0^{\frac{y}{\sin\varphi_0}} \frac{r\,dr}{\sqrt{\frac{2r}{a_0}}} \int_0^{\sqrt{\frac{2a_0\gamma_1}{r}}} d\varphi' \int_0^{\gamma_1 - \frac{r}{a_0}\frac{\varphi'^2}{2}} \frac{\zeta^{2\lambda-1}\,d\zeta}{\left(\gamma_1 - \frac{r}{a_0}\frac{\varphi'^2}{2} - \zeta\right)^{\frac{1}{2}}}, \qquad y' = \varphi - \varphi_0. \tag{4.50}$$

The substitution $\zeta = \left(\gamma_1 - \frac{r}{a_0}\frac{\varphi'^2}{2}\right) y_2$ /23/ yields the expression

$$\left(\gamma_1 - \frac{r}{a_0}\frac{\varphi'^2}{2}\right)^{2\lambda-\frac{1}{2}} \int_0^1 \frac{y_2^{2\lambda-1}\,dy_2}{(1-y_2)^{\frac{1}{2}}} \tag{4.51}$$

for the inner integral. After calculating the integrals and passing to the limit $\lambda \to 0$, the value of (4.50) can be found, taking into account (4.46), in the form

$$\frac{1}{2} a_0 \int_0^{\frac{y}{\sin\varphi_0}} \Big(a_0^2 \frac{\partial\tau_1}{\partial x'}\frac{\partial t_1}{\partial x'} + a_0^2\frac{\partial\tau_1}{\partial y'}\frac{\partial t_1}{\partial y'} - \frac{n+1}{4}\frac{f_1}{a_0^2}\Big) dr, \tag{4.52}$$

where the derivatives are taken along the straight line $\varphi=\varphi_0$. The condition $\varphi_2=0$ on the shock front is replaced by the requirement $\varphi_2=0$ on $u=\tau_1$ (obviously, $\varphi_1=0$ on $u=\tau_1$ from (4.39)). The equation for $t_1(x, y)$ is then

$$a_0^2\frac{\partial\tau_1}{\partial x}\frac{\partial t_1}{\partial x}+a_0^2\frac{\partial\tau_1}{\partial y}\frac{\partial t_1}{\partial y}-\frac{n-1}{4}\frac{f_1}{a_0^2}=0. \tag{4.53}$$

It was shown above that it agrees with the equation of the shock wave AB (Figure 5), i.e., $t=\tau_1+\gamma t_1$. Thus, if we take $\gamma t_1(x, y)$ in (4.34) as the adjustment to the linear equation of the shock wave $t=\tau_1$, the solution (4.41) will satisfy the condition that the potential φ vanishes on the shock wave

$$t=\tau_1+\gamma t_1. \tag{4.54}$$

The integrals of $\zeta^{2\lambda}$ in (4.48) reduce for $R'(t)=\text{const}$ to the form

$$\frac{1}{2\pi a_0^2}2\int_0^{\frac{y}{\sin\varphi_0}}\frac{rdr}{\sqrt{\frac{2r}{a_0}}}\int_0^{\sqrt{\frac{2a_0\tau_1}{r}}}d\varphi'\left(\gamma_1-\frac{r}{a_0}\frac{\varphi'^2}{2}\right)^{2\lambda+\frac{1}{2}}\int_0^1\frac{y_2^{2\lambda}dy_2}{(1-y_2)^{1/2}}. \tag{4.55}$$

If we differentiate (4.55) with respect to u or γ_1, set $\lambda=0$, and take into account that the integral of ζ^{λ} yields the same value in the limit as the integral of $\zeta^{2\lambda}$, then

$$\frac{\partial\varphi_2}{\partial u}=\frac{1}{\pi a_0^2}\int_0^{\frac{y}{\sin\varphi_0}}rdr\int_0^{\sqrt{\frac{2a_0\tau_1}{r}}}\frac{\Phi(x', y')}{\left(2\frac{r\gamma_1}{a_0}-\frac{r^2}{a_0^2}\varphi'^2\right)^{1/2}}, \tag{4.56}$$

where $\Phi(x', y')$ represents f for $\zeta=1$ in (4.48), except for the first term.

Finally

$$\frac{\partial\varphi_2}{\partial u}=\frac{1}{2}\int_0^{\frac{y}{\sin\varphi_0}}\frac{1}{a_0}\Phi(x-r\cos\varphi_0,\ y-r\sin\varphi_0)\,dr \tag{4.57}$$

on AB, the pressure being

$$P_2=-\gamma^2\rho_0\frac{\partial\varphi_2}{\partial u}. \tag{4.58}$$

Consider the case $R'(t)=V$, $P_\Phi(t)=P_1^0$. The equation of the linear shock wave (1.3) is then given by

$$\tau_1=\frac{x}{a_0}\cos\varphi_0+\frac{y}{a_0}\sin\varphi_0, \tag{4.59}$$

the adjustment to the equation of the shock wave (2.5) is

$$t_1=\frac{y}{\operatorname{tg}\alpha_1}\frac{n+1}{4}\frac{1}{a_0}\sin\alpha_1\frac{1}{\cos^2\alpha_1},\quad \sin\alpha_1=\cos\varphi_0, \tag{4.60}$$

and the potential in the linear problem is

$$\gamma\varphi_1=-\frac{P_1^0}{\rho_0}(u-\tau_1)+\frac{P_1^0}{\rho_0}\frac{P_1^0P_a'(1)\frac{a_0\sin^2\varphi_0}{\cos\varphi_0}}{x\sin^2\varphi_0-y\sin\varphi_0\cos\varphi_0}\frac{1}{2}(u-\tau_1)^2. \tag{4.61}$$

From (4.57) the pressure is easily found to be

$$P_2 = -\frac{n+1}{4}\frac{P_1^0}{\rho_0 a_0^2} P_1^0 P_a'(1)\frac{V-\xi}{\xi - a_0 \sin\alpha_1}\frac{1}{\sin\alpha_1}, \quad \xi = \frac{x}{t}, \tag{4.62}$$

which agrees with the simple-wave solution in § 2.

For arbitrary $R(t)$, the region of integration with respect to r and φ in the integral over U (cf. (4.45)) is

$$r(\varphi - \varphi_0) = \pm\sqrt{\frac{2a_0\gamma_1 r}{1 - \frac{r\sin^2\alpha_1 R''}{Aa_0}}},$$

$$A = \frac{a_0\cos^2\alpha_1}{\sin\alpha_1} - a_0 R''\frac{t - \tilde{t}_0 - \frac{r}{a_0}}{R'^2}, \quad \frac{A}{A - \frac{r}{a_0}\sin^2\alpha_1 R''} =$$

$$= \frac{R_0(r)}{R_0(0)}, \quad R_0(r) = t - \tilde{t}_0 - \frac{r}{a_0} - \frac{R'^2\cos^2\alpha_1}{a_0^2 R''}. \tag{4.63}$$

where $\frac{1}{a_0 R_0(0)}$ is the curvature of AB_1.

The radicand in the integral over U in (4.41) can be written as follows in terms of γ_1:

$$(t'-t)^2 - \frac{r^2}{a_0^2} = 2\gamma_1\frac{r}{a_0} - \frac{r^2}{a_0^2}\left(1 - \frac{r}{Aa_0}\sin^2\alpha_1 R''\right)(\varphi-\varphi_0)^2 - 2\frac{r}{a_0}\zeta. \tag{4.64}$$

Now it can be shown that (4.57) is still valid if the factor $\left(\frac{R_0(r)}{R_0(0)}\right)^{\frac{1}{2}}$ is added to the integrand. In the axisymmetric problem for the pressure near the front, one should use the factor $\sqrt{\frac{x'}{x}}$ in the integrand and adjust the function Φ to

$$\frac{n-1}{\rho_0 a_0^2}\frac{P_1}{\gamma}\frac{1}{x}\frac{\partial\varphi_1}{\partial x} - a_0\frac{1}{x}\frac{\partial\varphi_1}{\partial x}\frac{\partial t_1}{\partial x};$$

for constant $P_\Phi = P_1^0$, the linear solution is given in the form

$$\gamma\varphi_1 = -\frac{1}{\rho_0}A_0(u-\tau_1) - \frac{1}{2}\frac{1}{\rho_0}A_1(u-\tau_1)^2, \tag{4.65}$$

where $A_0 = f_1$ and $A_1 = f_2$ are given by (5.8) and (5.9). The solution on the front in the second approximation is

$$A_0 = \sqrt{\frac{Vt_0}{x}}P_1^0, \quad P_1 = A_0(x, y),$$

$$\lambda^0 = 2\frac{n-1}{n+1}\frac{a_0}{x - r\cos\varphi_0}A_0^2\sin\alpha_1 - \frac{2}{n+1}\frac{a_0^2\frac{\partial t_1}{\partial x}}{x - r\cos\varphi_0}A_0 P_1^0\sin\alpha_1,$$

$$P_2 = -\frac{n+1}{2}\rho_0\int_0^{\frac{y}{\sin\varphi_0}}\frac{\lambda^0 - A_0(x - r\cos\varphi_0, y - r\sin\varphi_0)A_1}{2\rho_0^2 a_0^3}\sqrt{\frac{x - r\cos\varphi_0}{x}}\,dr. \tag{4.66}$$

The pressure to second order on AB can also be found more simply. If, in virtue of $P_2 = -\rho_0 \frac{\partial \varphi_2}{\partial t}$, we set $\varphi_2 = -\frac{P_2}{\rho_0} D(u - \tau_1)$ in (4.35), where P_2 is the pressure to second order on the shock and f is given by (4.48), then terms on the left-hand side of (4.35) containing $\frac{\partial^2 D}{\partial \tau_1^2}$ cancel; if the left- and right-hand sides are equated, terms with $\frac{\partial D}{\partial \tau_1}$ give

$$2\frac{a_0^2}{\rho_0}\frac{\partial \tau}{\partial x}\frac{\partial P_2}{\partial x} + 2\frac{a_0^2}{\rho_0}\frac{\partial \tau}{\partial y}\frac{\partial P_2}{\partial y} + \frac{a_0^2}{\rho_0} P_2 \Delta\tau = +\Phi,$$

where Φ is the value of f for $\lambda = 0$. Since $2\frac{\partial \tau}{\partial x}\frac{\partial P_2}{\partial x} + 2\frac{\partial \tau}{\partial y}\frac{\partial P_2}{\partial y} = -2\frac{dP_2}{dt}\frac{1}{a_0^2}$ along rays (/27/, p. 114), and $\Delta\tau = -\frac{1}{a_0^3}\frac{1}{t - \bar{t}_0 - R'^3\cos^2\alpha_1/a_0^2 R''}$ (/27/, p. 116), by integrating along the ray $\bar{t}_0 = \text{const}$ we easily obtain

$$P_2 = -\frac{1}{2}\int_{\bar{t}_0}^{t} \Phi \sqrt{\frac{R_0(r)}{R_0(0)}}\, dt'.$$

which agrees with (4.56) if $t' = t - \frac{r}{a_0}$ is taken into account*.

We now return to the vicinity of the point B and use Whitham's method /10/ to determine the pressure along the shock wave. We introduce for the shock wave, as for an acoustic wave, a coordinate network $\alpha = a_0 t$, β, where $\beta = \text{const}$ is the equation of the rays normal to the front.

The length of the element BC (Figure 7) is $Ad\beta$ and the length of MB is $Md\alpha$, where A is the radius of curvature of the front and M is the ratio of the shock wave velocity to a_0.

The geometry of the motion yields the equations /10/

$$\frac{\partial \theta}{\partial \beta} = \frac{A'(M)}{M}\frac{\partial M}{\partial \alpha}, \qquad \frac{\partial \theta}{\partial \alpha} = -\frac{1}{A(M)}\frac{\partial M}{\partial \beta}, \tag{4.67}$$

where θ is the angle between the ray and the $0x$ axis, and $A(M)$ is given by the formula for the front along the ray in the linear problem /10/, namely

$$\frac{A}{A_0} = \left(\frac{M_0 - 1}{M - 1}\right)^2, \tag{4.68}$$

where the number M is related to the pressure by the expression $M - 1 = \frac{P}{Bn}\frac{n+1}{4}$.

Equations (4.67) give solutions with straight characteristics, i.e., simple waves, given by

$$\theta - \theta_1 = -\int_{M_0}^{M} \frac{dM}{\sqrt{-\frac{MA}{A'}}}; \quad \frac{A}{A'} = -\frac{2}{M-1} \tag{4.69}$$

* The terms in Φ, containing $\delta(u - \tau_1)$, vanish after the integration in virtue of (4.44).

or

$$\theta - \theta_1 = -2\sqrt{2}\ (\sqrt{M-1} - \sqrt{M_0 - 1}), \tag{4.70}$$

the characteristics being

$$\beta_1 = \frac{\beta}{\alpha} = c(M) = \sqrt{\frac{-M}{AA'}}.$$

From (4.70) for known $\theta_1 = \theta_B = \theta_0 - \sqrt{\frac{n+1}{2}\frac{P_1^0}{Bn}}$ and $M_0 = 1 + \frac{n+1}{4}\frac{P_1^0}{Bn}$, M and P can be found for any angle θ. Calculations by formula (4.70) showed that it gives a large pressure drop with increasing θ, and only for very small $\gamma = \frac{P_1^0}{Bn}$ can it be matched with solution (3.6). For example, when $M = 2$, $\frac{P_1^0}{Bn} = \frac{1}{200}$ we have from (3.6) values of $\frac{P}{P_1^0}$ of the order of 0.1 and matching is possible at a point on BB' with $\theta = 65°$.

The parabola (4.70) gives qualitatively the same results as depicted in Figure 6.

§5. Solution of the axisymmetric problem

The solution of the linear axisymmetric problem is given in /27/, and is derived from the linear solution of the general unsteady problem of the motion of a half-space subjected to a given pressure. The solution is

$$P = -\frac{1}{\pi}\frac{\partial}{\partial y}\int_0^{\psi_1} d\psi \int_{r_2^*}^{r_1^*} \frac{P_1^0 P_a\left(\frac{r}{Vt'}\right) r\,dr}{\sqrt{r^2 + x^2 - 2rx\cos\psi + y^2}} \tag{5.1}$$

for the hyperbolic region, where

$$t' = t - \frac{1}{a_0}\sqrt{r^2 + x^2 - 2rx\cos\psi + y^2},$$

r_1^*, r_2^* being the roots of the equation $r = Vt'$ for $r_1^*(\psi_1) = r_2^*(\psi_1)$ /27/.

For the elliptic region one should set $r_2^* = 0$, $\psi_1 = \pi$ in (5.1).

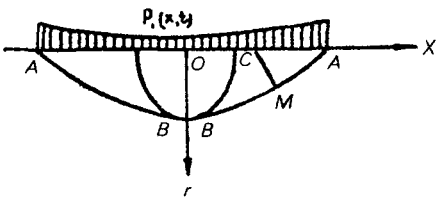


FIGURE 8.

To determine the pressure in the hyperbolic region ABC and on the shock wave AB to second order the method of §4 is used. If we denote $\frac{\varphi}{t} = \Phi$ and

express Φ and the coordinate η in a series form in γ, where

$$\Phi = \varphi_0 + \varphi_1, \quad \eta = \eta_0 + \eta_1(\xi_0), \quad \xi = \xi_0, \tag{5.2}$$

then φ_0 obeys the usual linear equation, and φ_1 is governed by

$$\frac{\partial^2\varphi_1}{\partial\xi_0^2}(\xi_0^2 - a_0^2) + 2\frac{\partial^2\varphi_1}{\partial\xi_0\partial\eta_0}\xi_0\eta_0 + \frac{\partial^2\varphi_1}{\partial\eta_0^2}(\eta_0^2 - a_0^2) - \frac{a_0^2}{\xi_0}\frac{\partial\varphi_1}{\partial\xi_0} =$$
$$= \frac{\partial^2\varphi_0}{\partial\xi_0^2}\left\{(n-1)\xi_0\frac{\partial\varphi_0}{\partial\xi_0} - (n-1)\varphi_0 + (n-1)\eta_0\frac{\partial\varphi_0}{\partial\eta_0}\right\} +$$
$$+ \frac{\partial^2\varphi_0}{\partial\xi_0\partial\eta_0}\left\{2\frac{d\eta_1}{d\xi_0}(\xi_0^2 - a_0^2) + 2\frac{\partial\varphi_0}{\partial\xi_0}\eta_0 - 2\left(\eta_1 - \frac{\partial\varphi_0}{\partial\eta_0}\right)\xi_0\right\} +$$
$$+ \frac{\partial^2\varphi_0}{\partial\eta_0^2}\left\{2\frac{d\eta_1}{d\xi_0}\xi_0\eta_0 - 2\eta_0\eta_1 + (n+1)\eta_0\frac{\partial\varphi_0}{\partial\eta_0} - (n-1)\varphi_0 + (n-\right.$$
$$\left. - 1)\xi_0\frac{\partial\varphi_0}{\partial\xi_0}\right\} + \frac{\partial\varphi_0}{\partial\xi_0}\frac{1}{\xi_0}(n-1)\left(\xi_0\frac{\partial\varphi_0}{\partial\xi_0} + \eta_0\frac{\partial\varphi_0}{\partial\eta_0} - \varphi_0\right) +$$
$$+ \frac{\partial\varphi_0}{\partial\eta_0}\left\{\frac{d^2\eta_1}{d\xi_0^2}(\xi_0^2 - a_0^2) - \frac{a_0^2}{\xi_0}\frac{d\eta_1}{d\xi_0}\right\}. \tag{5.3}$$

The quantity φ_0 obeys the following equation:

$$\frac{\partial^2\varphi_0}{\partial\xi_0^2}(\xi_0^2 - a_0^2) + 2\frac{\partial^2\varphi_0}{\partial\xi_0\partial\eta_0}\xi_0\eta_0 + \frac{\partial^2\varphi_0}{\partial\eta_0^2}(\eta_0^2 - a_0^2) - \frac{a_0^2}{\xi_0}\frac{\partial\varphi_0}{\partial\xi_0} = 0 \tag{5.4}$$

and to solve it for φ_0 in the vicinity of AB we use the following substitution in the form of a ray series:

$$\frac{\rho_0 a_0\varphi_0}{\cos\alpha_1} = \sum_{k=1}^{\infty} f_k(\xi_0)\bar{\eta}^k; \quad \bar{\eta} = \eta_0 - V\operatorname{tg}\alpha_1 + \zeta_0\operatorname{tg}\alpha_1, \tag{5.5}$$

where $\eta_0 = (V - \xi_0)\operatorname{tg}\alpha_1$ is the equation of AB in the linear case.

Substituting (5.5) in (5.4) and equating the coefficients of zero and first powers of $\bar{\eta}$, we obtain

$$2\frac{df_1}{d\xi_0}(\xi_0 V - a_0^2) - \frac{a_0^2}{\xi_0}f_1 = 0, \tag{5.6}$$

$$2\frac{df_2}{d\xi_0}(\xi_0 V - a_0^2) - \frac{a_0^2}{\xi_0}f_2 + \operatorname{ctg}\alpha_1\frac{d^2f_1}{d\xi_0^2}(\xi_0^2 - a_0^2) = 0. \tag{5.7}$$

The solution has the form

$$f_1 = P_1^0\frac{1}{\cos\alpha_1}\sqrt{\frac{\xi_0 - a_0\sin\alpha_1}{\xi_0}}, \tag{5.8}$$

$$f_2 = f_1\cos\alpha_1\sin\alpha_1\frac{1}{8}\int_{\xi_0}^{\xi_*}\frac{(4\xi_0 - 3a_0\sin\alpha_1)(\xi_0^2 - a_0^2)\,d\xi_0}{\xi_0^2(\xi_0 - a_0\sin\alpha_1)^3}, \tag{5.9}$$

i.e., a tabulated integral.

The solution of (5.3) is now sought in the form

$$\frac{\rho_0 a_0\varphi_1}{\cos\alpha_1} = P_2\bar{\eta}.$$

If this expression is substituted in the left-hand side of (5.3) and expression (5.9) in the right-hand side of it, an equation for P_2 is derived in the form

$$(\xi_0 - a_0 \sin\alpha_1)\frac{dP_2}{d\xi_0} - \frac{1}{2}\frac{a_0 \sin\alpha_1 P_2}{\xi_0} = \frac{1}{2}f, \tag{5.10}$$

where f represents the right-hand side of (5.3). The solution of (5.10) is

$$P_2 = \sqrt{\frac{\xi_0 - a_0\sin\alpha_1}{\xi_0}}\,\frac{1}{2}\int_{\xi_0}^{\xi_2} \frac{f\sqrt{\dfrac{\xi_0}{\xi_0 - a_0\sin\alpha_1}}}{a_0(\xi_0 - a_0\sin\alpha_1)}\,\alpha\xi_0. \tag{5.11}$$

When the terms in f containing $\delta(\tilde{\eta})$ (the delta function) are equated to zero, i.e., applying the condition that φ_2 vanishes on the front AB, the following equation is derived for η_1:

$$\frac{d\eta_1}{d\xi_0}(\xi_0 - a_0\sin\alpha_1) - \eta_1 = -\frac{n+1}{4}\frac{f_1}{p_0 a_0\cos\alpha_1},$$

which agrees with the equation of the shock wave AB.

Its solution gives the equation of AB to first order, namely

$$\eta = (V - \xi_0)\operatorname{tg}\alpha_1 + \eta_1,$$

$$\eta_1 = \frac{P_1^0}{Bn}(\xi_0 - a_0\sin\alpha_1)\frac{\frac{n+1}{2}}{\sin\alpha_1\cos^2\alpha_1}\left(\sqrt{\frac{\xi_0}{\xi_0 - a_0\sin\alpha_1}} - \frac{1}{\cos\alpha_1}\right). \tag{5.12}$$

In (5.11) the function f (according to (5.3) and (5.8), (5.9)) has the form

$$f = \frac{n+1}{2}\frac{a_0}{\cos^3\alpha_1}f_1 f_2 + 2\frac{df_1}{a\xi_0}\left\{\left(\operatorname{tg}^2\alpha_1\,\xi_0 + na_0\frac{\sin\alpha_1}{\cos^2\alpha_1} + \xi_0\right)f_1 + \frac{d\eta_1}{d\xi_0}(\xi_0^2 - a_0^2) - \eta_1\xi_0\right\} +$$
$$+ f_1^2\frac{1}{\xi_0}(n-1)\frac{a_0\sin\alpha_1}{\cos^2\alpha_1} + f_1\frac{d^2\eta_1}{d\xi_0^2}(\xi_0^2 - a_0^2). \tag{5.13}$$

The solution on the shock wave AB to second order is thus given by (5.11) and (5.13).

In the two-dimensional problem of § 2, equation (5.3) contains only terms with second derivatives. Substituting the linear solution of § 1 (written in terms of ξ_0, η_0) in the right-hand side of (5.3) for the two-dimensional case and using the fact that according to § 2

$$\eta_1 = \frac{n+1}{4}\frac{P_1^0}{Bn}\frac{1}{\cos^2\alpha_1}(V - \xi_0)\operatorname{tg}\alpha_1, \tag{5.14}$$

we obtain the right-hand side of (5.3) in the form

$$P_1^0 P_a'(\zeta_0')\frac{\operatorname{tg}\psi}{\xi_0\sin\psi - \eta_0\cos\psi}\left\{\frac{(n+1)a_0}{\cos\psi} + 2a_0\int_{\psi_0}^{\psi}\frac{P_1^0 P_a}{Bn}\sin\psi d\psi - \right.$$
$$\left. - 2\frac{\sin\psi}{\cos\psi}a_0\int_{\psi_0}^{\psi}\frac{P_1^0 P_a}{Bn}\cos\psi d\psi + \frac{n+1}{2}\frac{\operatorname{tg}\alpha_1}{\cos^2\alpha_1}\sin\psi\left(a_0 - \frac{V}{\cos\psi}\right)\right\}, \tag{5.15}$$

where

$$\xi_0\cos\psi + \eta_0\sin\psi = a_0.$$

We now show that the solution of (5.3) for the two-dimensional case is given by the simple-wave theory to second order. According to § 2

$$\frac{\partial V_x}{\partial \xi} = \frac{1}{\rho a} P_1^0 P_a'(\xi') \frac{\partial \xi'}{\partial \xi} \cos \varphi, \quad \frac{\partial V_x}{\partial \eta} = \frac{\partial V_x}{\partial \xi} \frac{\sin \varphi}{\cos \varphi},$$

$$\frac{\partial V_y}{\partial \eta} = \frac{\partial V_x}{\partial \xi} \frac{\sin^2 \varphi}{\cos^2 \varphi},$$

where $\varphi = \text{const}$ is a characteristic; hence, in terms of

$$\xi = \xi_0, \quad \eta = \eta_0 + \eta_1(\xi_0), \quad V_x = \frac{\partial \overline{\varphi}}{\partial \xi}, \quad V_y = \frac{\partial \overline{\varphi}}{\partial \eta},$$

we obtain

$$\frac{\partial^2 \varphi_1}{\partial \xi_0^2} = \frac{\partial V_x}{\partial \xi} - \frac{\partial^2 \varphi_0}{\partial \xi_0^2} + 2 \frac{\partial^2 \overline{\varphi}_0}{\partial \xi_0 \partial \eta_0} \frac{d\eta_1}{d\xi_0}, \tag{5.16}$$

$$\frac{\partial^2 \varphi_1}{\partial \xi_0 \partial \eta_0} = \frac{\partial V_x}{\partial \eta} - \frac{\partial^2 \varphi_0}{\partial \xi_0 \partial \eta_0} + \frac{\partial^2 \varphi_0}{\partial \eta_0^2} \frac{d\eta_1}{d\xi_0}, \quad \frac{\partial^2 \varphi_1}{\partial \eta_0^2} = \frac{\partial V_y}{\partial \eta} - \frac{\partial^2 \overline{\varphi}_0}{\partial \eta_0^2}.$$

Consider now the nonlinear case. If $\varphi = \psi + \varphi_2$ in formulas (2.1) they become

$$\varphi_2 = \frac{V_x \cos \psi + V_y \sin \psi + \dfrac{n-1}{2} \dfrac{P_1^0 P_a}{Bn} - \eta_1 \sin \psi}{\eta_0 \cos \psi - \xi_0 \sin \psi}.$$

Substituting (5.16) in the left-hand side of (5.3) for the two-dimensional case, a nonzero result is given by the terms which are obtained from those in (5.16) containing the factor $\frac{d\eta_1}{d\xi_0}$, or from terms obtained from the expansion of $\cos \varphi$ and $\sin \varphi$ in φ_2; the indicated result agrees with (5.15). Thus, the solution of the nonlinear problem of § 2 to second order by the simple-wave method can be obtained by the general method of solving equation (5.3) for the two-dimensional case.

Values of the pressure on AB when $\frac{V}{a_0} = 2$, $P_1^0 = 1000$ atm, $P_a(\xi) = 7/8 \frac{\xi}{V} + 1.8$ are given in the following table.

$\frac{\xi}{V}$	$P_1 = P_{\text{acoustic}}$	P
0.9	981.26	910.97
0.8	957.48	794.74
0.7	925.84	635.47
0.6	881.99	403.11
0.5	816.49	25.29

Let us now consider the segment of the shock wave BB' (Figure 5) which bounds the elliptic region.

In the linear case (5.1) indicates that

$$\frac{P}{P_1^0} = \varphi(\theta)\,\frac{t-\frac{r_1}{a_0}}{\frac{r_1}{a_0}} \tag{5.17}$$

near $r_1 = a_0 t$ /27/, where

$$\varphi(\theta) = 2\sin\theta\, M^2 \int_0^1 P_a(\xi)\,\xi\,\frac{\frac{\alpha}{2} + 2(1-\alpha^2)\frac{1}{4\alpha^2} + \frac{(1-\alpha^2)^2\xi}{16\alpha^5}}{(1+M\xi\cos\theta)^3}\,d\xi,$$

$$\alpha = \sqrt{\frac{1-M\xi\cos\theta}{1+M\xi\cos\theta}}, \quad M = \frac{V}{a_0}.$$

When $P_a \equiv 1$ then

$$\varphi(\theta) = \frac{M^2\sin\theta}{(1-M^2\cos^2\theta)^{3/2}}. \tag{5.18}$$

As before, the linear characteristics are replaced by the improved ones ($t-\frac{r_1}{a_0} \to y_1$, where y_1=const are the improved characteristics). Using (5.17), equation (3.2) for the characteristics takes the following form, which in the case of finite $\theta-\theta_0$ will have one degree of freedom in r_1:

$$t = \alpha(r_1) - \varphi(\theta)\frac{n+1}{2}\,y_1\frac{P_1^0}{Bn}\ln\frac{r_1}{a_0 y_1} + y_1, \quad \alpha(r_1) = \frac{r_1}{a_0}. \tag{5.19}$$

The reciprocal of the shock wave velocity will be

$$\frac{t}{r_1} = \frac{1}{a_0} - \frac{n+1}{4}\,\frac{P_1^0}{Bn}\,\varphi(\theta)\,\frac{y_1}{r_1}. \tag{5.20}$$

From (5.19) and (5.20) it is seen that on the shock wave

$$\frac{P}{P_1^0} = \varphi(\theta)\,e^{-\frac{1}{\frac{P_1^0}{Bn}\frac{n+1}{2}\varphi(\theta)}}. \tag{5.21}$$

Thus, in contrast to the two-dimensional case, we have an exponential order of smallness.

Relation (5.21) is valid on BB' everywhere, except the vicinity of the point B (Figure 5). Near the point B, $\theta-\theta_0$ is small and $\varphi(\theta)$ is unbounded, so that close to $\theta=\theta_0$ (5.18) shows that

$$\varphi(\theta) = \frac{\varphi_1(\theta_0)}{(\theta-\theta_0)^{3/2}}, \quad \varphi_1(\theta_0) = \frac{1}{2^{3/2}\cos^{5/2}\theta_0\sin^{1/2}\theta_0}. \tag{5.22}$$

Thus, near the point B the solution is (5.21), where $\frac{P_1^0}{Bn}\varphi(\theta)$ is of the order of $\frac{P_1^0/Bn}{(\theta-\theta_0)^{3/2}}$. For $\theta-\theta_0 \gg \left(\frac{P_1^0}{Bn}\right)^{\frac{2}{3}}$ this ratio remains small and (5.21) holds. If $\theta-\theta_0 \sim (P_1^0/Bn)^{\alpha}$, where $\alpha < \frac{2}{3}$, (5.21) yields an exponential order of P/Bn; when $\theta-\theta_0 \sim (P_1^0/Bn)^{\frac{2}{3}}\cdot\alpha_1\ln\frac{P_1^0}{Bn}$, $\alpha_1 < 0$, the order of smallness is a power of P/Bn.

In the case $\theta - \theta_0 \sim \left(\frac{P_1^0}{B\pi}\right)^{\frac{2}{3}}$, the motion will be two-dimensional and in (3.2) derivatives with respect to θ cannot be discarded.

Investigating the general relation (5.1) for the pressure in the whole neighborhood of B (Figure 5), the same result is obtained.

From (5.1), for small $t - \frac{r_1}{a_0} = y_1$, it can be derived that

$$P = \frac{1}{\pi} P_1^0 \frac{1}{r_1} \times$$

$$\times \int_0^{\pi} \frac{\frac{a_0 t}{M} - x\cos\psi - \sqrt{\left(\frac{a_0 t}{M} - x\cos\psi\right)^2 - (a_0^2 t^2 - x^2 - y^2)\left(\frac{1}{M^2} - 1\right)}}{\left(\frac{1}{M^2} - 1\right)\sqrt{\left(\frac{a_0 t}{M} - x\cos\psi\right)^2 - (a_0^2 t^2 - x^2 - y^2)\left(\frac{1}{M^2} - 1\right)}} d\psi \quad (5.23)$$

or

$$P = \frac{1}{\pi} P_1^0 \frac{1}{r_1} \times$$

$$\int_0^{\pi} \frac{\frac{r_1}{M} - r_1\cos\theta\cos\psi + \frac{ay_1}{M} - \sqrt{\left(\frac{r_1}{M} - r_1\cos\theta\cos\psi + \frac{ay_1}{M}\right)^2 - (a_0^2 t^2 - r_1^2)\left(\frac{1}{M^2} - 1\right)}}{\left(\frac{1}{M^2} - 1\right)\sqrt{\left(\frac{r_1}{M} - r_1\cos\theta\cos\psi + \frac{ay_1}{M}\right)^2 - (a_0^2 t^2 - r_1^2)\left(\frac{1}{M^2} - 1\right)}} d\psi. \quad (5.24)$$

Let us find the solution on the ray $\cos\theta \approx \frac{1}{M}$, where higher-order small quantities are discarded. This yields

$$P = \frac{1}{\pi} P_1^0 \frac{1}{r_1} \times$$

$$\times \int_0^{\pi} \frac{2\sin^2\frac{\psi}{2} r_1\cos\theta + a_0 y_1 \frac{1}{M} - \sqrt{4\sin^4\frac{\psi}{2} r_1^2\cos^2\theta - y_1 a_0 2r_1\left(\frac{1}{M^2} - 1\right)}}{\left(\frac{1}{M^2} - 1\right)\sqrt{4\sin^4\frac{\psi}{2} r_1^2\cos^2\theta - y_1 a_0 2r_1\left(\frac{1}{M^2} - 1\right)}} d\psi. \quad (5.25)$$

In expression (5.25) a singularity appears near the point $\psi = 0$, and it can be easily seen that $\sin^4\frac{\psi}{2} \sim y_1$. Using this together with the substitution $\frac{\psi}{2} = \varphi$, $\sin^4\varphi = \frac{a_0 y_1 (1 - 1/M^2)t'}{2r_1\cos^2\theta} = y_1^0 t'$, and discarding higher-order small quantities, the pressure near the point $x = \frac{a_0 t}{M}$ is found to be

$$P = K\sqrt[4]{y_1^0}, \quad (5.26)$$

where

$$K = \frac{1}{2\pi} P_1^0 \sin\theta \int_0^{\infty} \frac{(\sqrt{t'} - \sqrt{t'+1})\, t'^{-\frac{3}{4}}}{-\left(1 - \frac{1}{M^2}\right)\sqrt{t'+1}} dt'.$$

When $r_1 > a_0 t$, (5.1) gives the pressure near the point B as

$$\frac{P}{P_1^0} = 2\,\frac{1}{\pi}\,\frac{1}{r_1}\int_0^{\psi_1} y\,\frac{\left(x-\frac{a_0 t}{M}-x\,\frac{\psi^2}{2}\right)}{\left(1-\frac{1}{M^2}\right)\sqrt{\left(x-\frac{a_0 t}{M}-x\,\frac{\psi^2}{2}\right)^2 - M^2(M^2-1)\,a_0^2 y_1 2t}}\,d\psi,$$

$$y_1 = r_1 - a_0 t,\quad \frac{\psi_1^2}{2} = \frac{x-\frac{a_0 t}{M}-\sqrt{\left(1-\frac{1}{M^2}\right)a_0^2 y_1 2t}}{x}. \tag{5.27}$$

If $x-\frac{a_0 t}{M} \sim \sqrt{y_1}$, $P \sim P_1^0 \sqrt[4]{y_1} \sim P_1^{0\frac{4}{3}}$, we have two-dimensional flow. For $x-\frac{a_0 t}{M} \gg \sqrt{y_1}$, $x-\frac{a_0 t}{M} \sim y_1^{\frac{\alpha}{2}}$ and $\alpha < 1$, (5.27) yields $P \sim P_1^0 \sqrt{x-\frac{a_0 t}{M}+\sqrt{y_1}}$. Formal application of the theory in this section gave the motion as one-dimensional in r_1; however the relations gave $y_1 < 0$, which is impossible. In the case of finite $x-\frac{a_0 t}{M}$ (5.27) yields (5.8), i.e., the one-dimensional solution along the rays (5.5). For $x-\frac{a_0 t}{M} \sim y_1^{\frac{\alpha}{2}}$ and $\alpha < 1$, a criterion* of one-dimensionality along the rays is that $P_1 \gg P_2$, where P_1 is the first approximation and P_2 is the second. From (4.66) or (5.16) we then have $x-\frac{a_0 t}{M} \gg P_1^{0\frac{2}{3}}$. Hence it follows that for $\alpha < 1$ and $P_1 \gg P_2$, (5.27) can be used for the pressure. If the pressure is taken according to (5.27) the shock wave (5.12) does not intersect $r_1 = (a+v_r)\,t$, but if the modified value $P = P_1^0 \sqrt{x-\frac{a_0 t}{M}+\sqrt{y_1}}$ is used, an intersection will exist to the order $x-\frac{a_0 t}{M} \sim P_1^{0\frac{2}{3}}$.

A solution in the vicinity of the point B (Figure 2) should be looked for in the form of short waves /35/. Consider the inequalities $\theta - \theta_0 \gg \left(\frac{P_1^0}{Bn}\right)^{1/3}$ $r_1 < a_0 t$. Expression (5.1) yields (similarly to (5.24))

$$P = P_1^0\,\frac{1}{\pi}\sin\theta\int_0^{\pi}\frac{y^{1/t}\,d\psi}{2\left(\cos\theta_0\sin^2\frac{\psi}{2}+\sin\theta_0\frac{\theta-\theta_0}{2}\right)^2}.$$

The substitution $\frac{2}{M}\,\frac{\sin^2\psi_2}{\theta-\theta_0} = t_1$ gives it the form

$$P = \frac{1}{\pi}\,P_1^0\sin\theta\,\frac{y_1}{t}\,(\theta-\theta_0)^{-\frac{3}{2}}\sqrt{2}\sqrt{M}\int_0^{\infty}\frac{dt_1}{\sqrt{t_1}\,(t_1+\sin\theta_0)^2} = P_1 K(\theta)\,\frac{y_1}{t}\,\frac{1}{(\theta-\theta_0)^{3/2}}. \tag{5.28}$$

* The exact condition of one-dimensionality is that the curvature of the shock wave is small:

$$\frac{d^2\eta}{d\xi^2} \sim \frac{P_1^0}{\left(x-\frac{a_0 t}{M}\right)^{1/2}},\quad x-\frac{a_0 t}{M} \gg P_1^{0\frac{2}{3}}.$$

If $\theta - \theta_0$ is allowed to approach the point B along the parabolas $\theta - \theta_0 = \left(\frac{y_1}{\frac{r_1}{a_0}}\right)^{\alpha}$, then on the shock wave

$$\frac{y_1}{\frac{r_1}{a_0}} = \left(1 + \frac{\frac{3}{2}\alpha}{2 - \frac{3}{2}\alpha} \frac{\frac{2}{n+1}\frac{Bn}{P_1^0 K}}{a_0}\right)^{-\frac{2}{3\alpha}}, \tag{5.29}$$

i.e., of lower order than exponential. As $\alpha \to 0$, (5.29) gives y_1 as an exponential function. Thus, when moving along BB' (Figure 5) for finite $\theta - \theta_0$ (5.21) holds, and the same solution is preserved when approaching B for the values $\theta - \theta_0 \gg \left(\frac{P_1^0}{Bn}\right)^{\frac{1}{3}}$. When $\theta - \theta_0 \sim \left(\frac{P_1^0}{Bn}\right)^{\frac{1}{3}}$ the solution is a two-dimensional short wave and can be matched to the previous one by means of (5.21).

Consider the relations for the region in which $\theta - \theta_0 \sim \left(\frac{P_1^0}{Bn}\right)^{\frac{2}{3}}$ /9/. If we set

$$v_r = a_0 M_0 \mu, \quad v_\theta = a_0 \sqrt{\frac{n+1}{2}} M_0^{\frac{3}{2}} v,$$

$$r_1 = a_0 t \left(1 + \frac{n+1}{2} M_0 \delta\right), \quad \theta - \theta_0 = \sqrt{\frac{n+1}{2} M_0}\, y,$$

the particular solution is /9/

$$\delta = -\frac{1}{2}\frac{d\varphi}{d\mu} y^2 + F(\mu),$$

$$v = \varphi y + v_0,$$

where

$$\varphi = \frac{A^2 - B^2 - \mu^2}{A + \mu},$$

$$F = \mu - \frac{(\mu + A)^2 - B^2}{2B} \ln \frac{\mu + A - B}{\mu + A + B} + C\left((\mu + A)^2 - B^2\right).$$

Near the $0y$ axis on the shock wave $\mu \to 0$, and therefore $A = B$. All the remaining constants are determined from the matching condition at the point B (Figure 5), $\delta \sim \mu$, and from the momentum conservation equation. However, a solution passing to the linear one $\mu = \delta^{\frac{1}{4}} f\left(\frac{y}{\sqrt{\delta}}\right)$, as in § 4, cannot be found. A more detailed consideration is given in /35/.

Equation (4.70), or the corresponding axisymmetric version /10/, gives a better approximation in the present case than in the two-dimensional problem owing to the greater pressure drop when approaching the $0y$ axis (Figure 12a).

§ 6. Validity of the Lighthill—Whitham method for the second approximation

The solutions of §§ 3 and 5 were derived from the linear solution taken along the improved characteristics. The relations on the shock wave will be

satisfied to second order, since the linear solution satisfies the shock relations in the first approximation and, according to (3.6), is a second-order small quantity on the shock wave. We show that the equations of motion are also satisfied to second order. The motion will be irrotational since an isentropic flow is assumed. Near the shock wave there is a pressure drop with a finite derivative with respect to r_1; therefore, the equations of motion will be those of short waves (equations (4.5)) in the two-dimensional case, i.e.,

$$(\mu - \delta)\frac{\partial \mu}{\partial \delta} + \frac{1}{2}\mu = 0, \tag{6.1}$$

where the variables $r_1 = a_0 t\left(1 + \frac{n+1}{2} M_0 \delta\right)$, $\frac{P}{Bn} = M_0 \mu$ are introduced.

Equation (6.1) is one dimensional and, in virtue of the relation $v_\theta \ll \frac{P}{\rho_0 a_0}(\theta - \theta_0)$, is satisfied by (3.1) up to the point B. In fact, after substituting (3.8) in the equations of gas dynamics (F) and using (3.1), it is found that the derivatives with respect to θ appear in the equations to a higher order than second and may be discarded. As the point B is approached the derivatives with respect to θ begin to be important when $v_\theta \approx \frac{P}{\rho_0 a_0}(\theta - \theta_0)$.

It must therefore be checked that solution (3.8) satisfies equation (6.1). It is clear that in (4.4) we may set

$$M_0 = \left(\frac{P_1^0}{Bn}\right)^2.$$

The solution of (6.1) has the form

$$\mu^2 = C(\delta - 2\mu), \tag{6.2}$$

and in our notation (3.3) is

$$a_0 t \frac{n+1}{2} \delta M_0 = a_0 \beta(r_1)\sqrt{y_1} - a_0 y_1.$$

If y_1 is replaced by P, (3.8) yields

$$\frac{n+1}{2}\left(\frac{P_1^0}{Bn}\right)^2 \delta = 2\frac{n+1}{2}\frac{P}{Bn} - a_0\left(\frac{P}{P_1^0}\right)^2 \frac{1}{a_0 f^2(\theta)}$$

or

$$\delta = 2\mu - \frac{2}{n+1}\frac{1}{f^2(\theta)}\mu^2, \tag{6.3}$$

which is the same as solution (6.2) with $C = -\frac{n+1}{2} f^2(\theta)$. Let us find C from the condition at the limiting wave. On $R = a_0$ equation (3.12) reduces to

$$-a_0 \frac{\partial \bar{\varphi}_1}{\partial R} = \frac{\partial^2 \bar{\varphi}_0}{\partial R^2}(n+1)a_0 \frac{\partial \bar{\varphi}_0}{\partial R}$$

or, in virtue of (3.1'), at $R = a_0$, $\frac{\partial \bar{\varphi}_1}{\partial R} = \frac{1}{2}\gamma^2 a_0 (n+1) f^2(\theta)$; but for $R = a_0$, $\frac{\partial \bar{\varphi}}{\partial R} = v_r$,

or in our notation

$$\frac{\partial \tilde{\tau}_1}{\partial R} = \gamma^2 \mu, \text{ when } \mu = \frac{n+1}{2} f^2(\theta) \text{ for } R = a_0.$$

By definition $r = R + r_2$, so that $\mu = \frac{n+1}{2} f^2(\theta)$ for $r = a_0 + \frac{n+1}{2} \gamma^2 \mu$, $\delta = \frac{n+1}{2} f^2(\theta)$, and $\delta - 2\mu = \frac{1}{C} \mu^2$, written for that point, gives $\mu = -C$, $C = -\frac{n+1}{2} f^2(\theta)$. Solution (3.8) thus satisfies equation (4.25) or the equations of gas dynamics (F), which in the present case reduce to the one-dimensional equation (6.1) and a condition at the point $r = a_0 + \left(\frac{n+1}{2}\right)^2 \gamma^2 f^2(\theta)$ with which to determine the constants. The values C found from equations (4.34) and (3.3) agree.

The same considerations can be applied to the segment of BB' (Figure 5) near the point B in the case of the inequality $\theta - \theta_0 \gg \left(\frac{P_1^0}{Bn}\right)^{\frac{1}{2}}$, solution (6.3) then being valid to the corresponding order.

We now show that the solution of the axisymmetric problem also satisfies the equations of gas dynamics.

For one-dimensional short waves in the axisymmetric problem we have /9/

$$(\mu - \delta) \frac{\partial \mu}{\partial \delta} + \mu = 0, \tag{6.4}$$

where

$$\frac{P}{Bn} = M_0 \mu, \quad r_1 = a_0 t \left(1 + \frac{n+1}{2} M_0 \delta\right).$$

Consider the solution (5.17) (where $t - \frac{r_1}{a_0} \to y_1$) and (5.19). The solution of equation (6.4) is

$$\delta = -\mu \ln \mu + C\mu. \tag{6.5}$$

From (5.19) we have

$$\frac{n+1}{2} M_0 \delta = \frac{y_1}{t} \frac{n+1}{2} \frac{P_1^0}{Bn} \varphi(\theta) \ln \frac{r_1}{a_0 y_1} - \frac{y_1}{t}, \tag{6.6}$$

whence, using (5.17),

$$\delta = \frac{P}{P_1^0} \frac{1}{M_0} \frac{P_1^0}{Bn} \ln \frac{P_1^0 \varphi(\theta)}{P} - \frac{P}{P_1^0 \varphi(\theta)} \tag{6.7}$$

or

$$\delta = \mu \ln \left(\frac{P_1^0}{Bn} \frac{\varphi}{M_0 \mu}\right) - \frac{M_0 \mu}{\frac{P_1^0}{Bn} \varphi(\theta)} = -\mu \ln \mu + \mu \ln \frac{P_1^0 \varphi(\theta)}{Bn M_0} - \mu \frac{M_0}{\frac{P_1^0}{Bn} \varphi(\theta)}. \tag{6.8}$$

If

$$C = \ln \frac{P_1^0 \varphi(\theta)}{Bn M_0} - \frac{M_0}{\frac{P_1^0}{Bn} \varphi(\theta)}, \quad M_0 = e^{-\frac{1}{\frac{P_1^0}{Bn} \frac{n+1}{2} \varphi(\theta)}},$$

equation (6.5) is satisfied.

Thus relations are derived showing the validity of applying Whitham's method to our problem.

Lighthill's technique was not applied rigorously when solving (3.2), and the proof given here is of the validity of his technique /3/ for the second approximation.

In addition, it is shown that when $\theta - \theta_0 \gg \left(\frac{P_1^0}{Bn}\right)^{\frac{1}{2}}$ in the two-dimensional problem and when $\theta - \theta_0 \gg \left(\frac{P_1^0}{Bn}\right)^{\frac{1}{3}}$ in the axisymmetric problem, the solution near the point B (Figure 5) will be correct to order α as follows:

for the two-dimensional problem $1 < \alpha \leqslant 2$,

for the axisymmetric problem $\frac{4}{3} < \alpha$.

The solutions (6.2) and (6.5) were obtained as exact solutions of the equations of gas dynamics by Sedov in 1945 /1/.

Further, near BC (Figure 2), one can obtain the pressure in the linear approximation of the axisymmetric problem in the form

$$\frac{P}{P_1^0} = C(\theta) + A\left(1 - \frac{r_1}{a_0 t}\right)\ln\left(1 - \frac{r_1}{a_0 t}\right), \quad r_1 < a_0 t,$$

$$\frac{P}{P_1^0} = C(\theta) + B\left(\frac{r_1}{a_0 t} - 1\right)\ln\left(\frac{r_1}{a_0 t} \quad 1\right), \quad r_1 > a_0 t$$

and from the relations on the shock wave $\frac{P}{P_1^0} - C(\theta)$ is of an exponential order of smallness in $\frac{P_1^0}{Bn}$.

The solution for the motion of a half-space due to a strong shock wave, only slightly differing from a wave of constant intensity, is given by the author in /30/.

Chapter II

PRESSURE DISTRIBUTION FOR ARBITRARY CONDITIONS AT THE BOUNDARY OF A HALF-SPACE

§1. Formulation of the problem

The boundary conditions (B) and (C) apply to an arbitrary pressure varying at the surface in time and space. Consider first the initial problem of an explosion above a liquid, for which we examine possible velocity magnitudes $R'(t)$ of the pressure front over the surface.

Let the velocity of the shock wave in the air be $D(t_2+t)$, and its radius $\int_0^{t_2+t} D(t')\,dt' = R_1(t_2+t)$. The radius of the shock front over the surface is then

$$R(t) = \sqrt{R_1^2(t_2+t) - h^2},$$

and its velocity

$$R'(t) = \frac{DR_1(t_2+t)}{\sqrt{R_1^2(t_2+t) - h^2}}, \tag{1.1}$$

where $h = R_1(t_2)$, and t is the time during which the shock pressure penetrates into the liquid. The time $t = \bar{t}$ at which the velocity $R'(t)$ equals the velocity of sound in the liquid, a, is given by

$$\frac{D(t_2+\bar{t})\,R_1(t_2+\bar{t})}{\sqrt{R_1^2(t_2+\bar{t}) - h^2}} = a. \tag{1.2}$$

Various cases are possible, depending on the form of the function $D(t_2+t)$. First suppose the velocity D is constant:

1) for $D \ll a$, equation (1.2) yields

$$1 - \frac{t_2^2}{(t_2+\bar{t})^2} = \frac{D^2}{a^2} \tag{1.3}$$

and from the condition $\frac{D}{a} \ll 1$, we obtain $\bar{t} < t_2$;

2) for $D = ka$, $k < 1$, we have

$$1 + \frac{\tilde{t}}{t_2} = \frac{1}{\sqrt{1 - k^2}}.$$

As $k \to 1$ it is clear that $\frac{\tilde{t}}{t_2} \to \infty$.

Now suppose $D(t_2 + t)$ varies. The following conditions for $\tilde{t}$ can be found from (1.2):

$$1)\ D(t_2 + t) \ll a, \int_{t_0}^{\tilde{t}} D dt' \ll \int_0^{t_2} D dt';$$

$$2)\ D(t_2 + t) = ka, \int_{t_0}^{\tilde{t}} D dt = \left(\frac{1}{\sqrt{1 - k^2}} - 1\right) \int_0^{t_2} D dt.$$

If $R_1 = A\sqrt{t_2 + t}$ (1.2) is replaced by

$$\frac{A^2}{2A\sqrt{\tilde{t}}} = a,\ \sqrt{\tilde{t}} = \frac{A}{2a},$$

and since $h = A\sqrt{t_2}$ the condition $\sqrt{\tilde{t}} \ll \sqrt{t_2}$ gives $A^2 \ll 2ah$.

Thus, for a given velocity $D(t_2 + t)$ the instant $t = \tilde{t}$ at which $R'(t)$ becomes subsonic can be determined. It is obvious from (1.2) that at $t = 0$, $R'(0) = \infty$.

Thus, in the general formulation of the problem for $h > 0$ two typical facts should be borne in mind.

First, since $R'(0) = \infty$ always the initial velocity of the pressure over the surface is infinite, and it will be shown that this results in a segment AB of the shock wave (Figure 5) reaching the $0y$ axis, and the absence of the segment BB' of the shock wave (Figure 5) which bounds the region of initial values. Consequently, all the features associated with this region are missing.

Second, in real conditions a time $t = \tilde{t}$ always exists at which the velocity $R'(\tilde{t})$ equals the velocity of sound a_0 or, more accurately, the perturbation velocity $\frac{a + v + a_0}{2}$. It will be shown that the shock front AB then leaves the boundary area and moves over the free surface of zero pressure.

The irregular reflection arising in this case near the surface requires separate consideration.

§2. Solution in the linear case and considerations regarding the solution of the nonlinear problem on the shock wave

Two problems will be considered simultaneously, namely, a problem with axial symmetry and a two-dimensional one, corresponding to a pressure along a straight line parallel to the surface at a height h.

In the axisymmetric case the $0x$ axis is chosen along the fluid surface, which in the linear formulation is considered to be nonvariable /27/, and the $0y$ axis is taken into the fluid.

The pressure obeys the wave equation

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{1}{x}\frac{\partial P}{\partial x} = \frac{1}{a_0^2}\frac{\partial^2 P}{\partial t^2}, \tag{2.1}$$

which must be solved subject to zero initial conditions and boundary conditions (B). Here a_0 is the initial velocity of sound in the fluid.

The solution of this problem can be found with the aid of retarded dipoles. If $R'(t) < a_0$, then the solution is described by (/27/, p. 31)

$$P = -\frac{1}{2\pi}\frac{\partial}{\partial y}\int_0^{2\pi} d\psi \int_0^{r_0} \frac{P_1(r, t')\, r dr}{\bar{R}_0}, \tag{2.2}$$

where $t' = t - \frac{\bar{R}_0}{a_0}$, $\bar{R}_0 = \sqrt{r^2 + x^2 - 2rx\cos\psi + y^2}$, and $r = r_0^*$ is the root of the equation

$$r = R(t'). \tag{2.3}$$

The disturbed region will be bounded by the sphere $r_1 = a_0 t$, on which the pressure vanishes.

It can be easily shown that in this case near the boundary $r_1 = a_0 t$ of the region (Figure 3), the solution (2.2) only involves the initial values $R'(0)$ and $P_\phi(0)$, and the pressure distribution near $r_1 = a_0 t$ is again given by (5.17) in Chapter I, where in $\varphi(\theta)$ we set

$$P_1^0 = P_\phi(0), \quad V' = R'(0).$$

The solution on the shock for finite t is then given by formula (5.21) in Chapter I.

Thus, the solution on the shock front in this case does not differ from that of constant pressure on the surface. For large t, an additional attenuation effect is present (see § 6, Chapter II).

The situation $R'(t) < a_0$ is only realized if the pressure arises at the surface and $h = 0$.

Suppose now $R'(t) > a_0$; the pressure in the fluid is then given by (2.2) for $r_1 = \sqrt{x^2 + y^2} \leqslant a_0 t$, and

$$P = -\frac{1}{2\pi}\frac{\partial}{\partial y}\int_{-\psi_1}^{\psi_1} d\psi \int_{r_2^*}^{r_1^*} \frac{P_1(r, t')\, r dr}{\bar{R}_0} \tag{2.4}$$

for $r_1 \geqslant a_0 t$. Here $r_{1,2}^*$ are the roots of equation (2.3) (which in this case has two roots), and ψ_1 is the boundary value of the angle ψ satisfied by the roots $r_1^* \geqslant r_2^*$.

The boundary of the disturbed region for finite $R'(0)$ is given by the curve AB (Figure 8), which is the envelope of spherical sound perturbations produced by a point A of the front in motion along the boundary at time $t_0 \leqslant t$, and by BB' (Figure 8), representing the boundary of the initial conditions on the shock wave $r_1 = a_0 t$.

On BB' the solution is again given by formula (5.17), with the exception of the vicinity of the point B, where the function $\varphi(\theta)$ has a singularity. In addition, one cannot restrict oneself to the initial values of P_Φ and R'.

In fact, (2.3) shows that near the point B (Figure 8)

$$r_0^* = R'(0)\left(y_1 - \frac{1}{a_0}\frac{r_0^{*2} - 2r_0^* x\cos\psi}{2a_0 t}\right) + \frac{1}{2}R''(0)\left(y_1 - \right.$$

$$\left. - \frac{1}{a_0}\frac{r_0^{*2} - 2r_0^* x\cos\psi}{2a_0 t}\right)^2,$$

where $y_1 = t - \frac{r_1}{a_0}$. If now $1 - \frac{R'(0)}{a_0}\frac{x}{a_0 t} \sim y_1^{\alpha}$ (where $\alpha < \frac{1}{2}$) then $r_0^* \sim y_1^{1-\alpha}$, and $r_0^{*2} \ll y_1$; the expression for r_0^* reduces to

$$r_0^* = \frac{R'(0)\, y_1}{1 - \frac{R'(0)}{a_0}\cos\theta\cos\psi}, \tag{2.5}$$

where $r_1\cos\theta = x$. For $\alpha = \frac{1}{2}$ the term $r_0^*\left(1 - \frac{R'}{a_0}\cos\theta\cos\psi\right)$ has the same order of smallness as $\frac{1}{2}R''(0)\frac{1}{a_0^2}\cos^2\theta\cos^2\psi\, r_0^{*2}$.

Near the point B the acceleration $R''(0)$ thus will affect the solution which will be a nonstationary short wave /9/. This case will not be considered since it corresponds to an unsteady pressure arising at the surface $h = 0$ and is physically more rarely encountered than the case $R'(0) = \infty$.

The equation of AB (Figure 8) is written as the envelope of elementary waves arising at time $t = t_0$ at the point $y = 0$, $x = R(t_0)$:

$$\left.\begin{aligned}(x - R)^2 + y^2 &= a_0^2 (t - t_0)^2\\ R'(t_0)(R - x) &= a_0^2 (t_0 - t)\end{aligned}\right\}, \tag{2.6}$$

where $R = R(t_0)$. At the point A (Figure 8), $t = t_0$, $x = R(t)$, $y = 0$; at the point B (Figure 8), t_0 0. When $R'(0) < \infty$ then

$$x_B = \frac{a_0^2 t}{R'(0)},$$

and when $R'(0) = \infty$, B lies on the 0y axis.

In the two-dimensional case the pressure distribution in the case $R'(t) < a_0$, or for the region $r_1 < a_0 t$ in the case $R'(t) > a_0$, is

$$P = -\frac{1}{2\pi}\frac{\partial}{\partial y}\int_{r_1'}^{\xi_2} d\xi \int_{-\eta_1'}^{\eta_1'} \frac{P_1(-\xi, t')}{\bar{R}_0}\, d\eta -$$

$$-\frac{1}{2\pi}\frac{\partial}{\partial y}\int_{\xi_1}^{\xi_2} d\xi \int_{-\eta_1}^{\eta_1} \frac{P_1(\xi, t')}{\bar{R}_0}\, d\eta, \qquad (2.7)$$

where $t' = t - \frac{1}{a_0}\bar{R}_0$, $\bar{R}_0 = \sqrt{(\xi - x)^2 + \eta^2 + y^2}$; η_1' and η_1 are defined by

$$f(-\xi) = t - \frac{1}{a_0}\sqrt{(\xi - x)^2 + \eta_1'^2 + y^2},$$

$$f(\xi) = t - \frac{1}{a_0}\sqrt{(\xi - x)^2 + \eta_1^2 + y^2},$$

$$r_1^* = -R\left(t - \frac{1}{a_0}\sqrt{r_1^{*2} - 2r_1^* x + x^2 + y^2}\right),$$

$$\xi_1 = R\left(t - \frac{1}{a_0}\sqrt{\xi_1^2 - 2\xi_1 x + x^2 + y^2}\right). \qquad (2.8)$$

where the integration limits correspond to the region given by the variables ξ, η, shown in Figure 9, and $t' = f(\xi)$ is the inverse function of $\xi = R(t)$.

In the region $r_1 > a_0 t$ only the second term on the right-hand side of expression (2.7) remains, so that

$$P = -\frac{1}{2\pi}\frac{\partial}{\partial y}\int_{\xi_1}^{\xi_2} d\xi \int_{-\eta_1}^{\eta_1} \frac{P_1(\xi, t')\, d\eta}{\bar{R}_0}, \qquad (2.9)$$

where $\xi_{1,2}$ are the two roots of equation (2.8).

In the case $R'(t) < a_0$ or $R'(t) > a_0$, but $r_1 < a_0 t$, and near BB' in Figure 3 or BB' in Figure 8 the pressure is again given by (3.1) in Chapter I, where $P_1^0 = P_\phi(0)$, $V = R'(0)$. The pressure distribution to second order is in this case given by (3.6) in Chapter I taken along the shock wave BB'. The shock wave in the linear formulation is given by (2.6) (AB in Figure 8).

For $R'(0) = \infty$, the coordinate x of the point B (Figure 8) lies on the $0y$ axis and the segment BB' of the initial conditions of the shock wave is missing.

In Chapter I a singularity along BB' near the point B arose in the linear solution, and this necessitated the replacement of the linear characteristics by improved ones and the introduction of short waves. These difficulties do not arise when $R'(0) = \infty$.

In this connection it only remains to show that the solution in the linear case is continuous at B.

It will be shown that both in the two-dimensional and in the axisymmetric problems the solutions agree at B when approaching this point along the shock wave AB and along the $0y$ axis.

First consider the two-dimensional problem. The pressure along the front AB is obtained using the ray representation to find the intensity of the wave fronts. At some point M of the front (see Figure 8) the pressure at time t is expressed in terms of the pressure at a point E of the ray ME, through which the front passes at time t_0. Hence

$$P_M = P_E \sqrt{\frac{r_E}{r_M}}, \tag{2.10}$$

where $r_{M,E}$ are the radii of curvature of the front at M, E, and $P_E = P_\phi(t_0)$. With the aid of (2.6) this yields (/27/, p. 118)

$$r_E = r_M(t_0); \quad \frac{r_M(t)}{a_0} = t - t_0 - \frac{R'^3(t_0)}{a_0^2 R''(t_0)} \sin^2\theta, \tag{2.11}$$

where $\sin^2\theta = 1 - \frac{a_0^2}{R'^2}$, and θ is the angle between the ray ME and the $0x$ axis.

For the axisymmetric case formula (2.10) contains a square root of the ratio of the radii of curvature at a section normal to the plane $x0y$:

$$P_M = P_E \sqrt{\frac{r_E}{r_M}} \sqrt{\frac{R(t_0)}{x}}. \tag{2.12}$$

Taking (2.10) along the front AB for the case defined by (1.1) and $D = \text{const}$, i. e.,

$$R(t) = D\sqrt{2t_2 t + t^2}, \tag{1.1'}$$

we derive

$$R'(t) = D\frac{t + t_2}{\sqrt{2t_2 t + t^2}}, \quad R''(t) = -D\frac{t_2^2}{(2t_2 t + t^2)^{3/2}}$$

and, for (2.11),

$$\frac{r_M(t)}{a_0} = t - t_0 + \frac{D^2(t_0 + t_2)^3}{a_0^2 t_2^2}\sin^2\theta, \quad \sin^2\theta = 1 - \frac{a_0^2}{D^2}\frac{2t_0 t + t_0^2}{(t_0 + t_2)^2},$$

where t_0 in (2.12) is found from (2.6).

Near the point B (1.1') gives the same solution as

$$R(t) = A_1\sqrt{t}, \quad A_1 = D\sqrt{2t_2}, \tag{2.13}$$

since $t_0 \to 0$ when approaching B (Figure 10).

At the point B in Figure 10 for the two-dimensional problem we have from (2.10)

$$\frac{P_B}{P_\phi(0)} = \sqrt{\frac{A_1^2/2a_0^2 t}{1 + A_1^2/2a_0^2 t}}. \tag{2.14}$$

Let us find the pressure at B (Figure 10) when approaching it along the $0y$ axis. The relation $P_1 = P_\phi(0)$ is assumed in (2.7); it is valid in the

vicinity of the point B. The pressure on the $0y$ axis is then

$$\frac{P}{P_\Phi(0)} = -\frac{2}{\pi}\frac{\partial}{\partial y}\int_0^{\sqrt{a_0^2 t^2 - x^2 - y^2}} d\eta \int_0^{r_0^*} \frac{d\xi}{\sqrt{\xi^2+\eta^2+y^2}}, \tag{2.15}$$

where

$$r_0^* = A_1\sqrt{t - \frac{y}{a_0} - \frac{r_0^{*2}}{2a_0 y} - \frac{\eta^2}{2a_0 y}} \tag{2.16}$$

near B. For small $a_0 t - y$ (2.15) becomes

$$\frac{P}{P_\Phi(0)} = -\frac{2}{\pi}\int_0^{\sqrt{a_0^2 t^2 - x^2 - y^2}} \frac{1}{y}\frac{\partial r_0^*}{\partial y}\, d\eta, \tag{2.17}$$

$$\frac{\partial r_0^*}{\partial y} = -\frac{1}{a_0}\sqrt{\frac{A_1^2}{1+\frac{A_1^2}{2a_0^2 y}}}\,\frac{1}{2\sqrt{t - \frac{y}{a_0} - \frac{\eta^2}{2a_0 y}}}, \tag{2.18}$$

which yields (2.14). Thus, in contrast to the case of a finite $R'(0)$, the point B (Figure 10) is not singular for the linear solution.

In the axisymmetric case near B (2.12) gives

$$\frac{P}{P_\Phi(0)} = \sqrt{\frac{A_1^2/2a_0^2 t}{1+A_1^2/2a_0^2 t}}\sqrt{\frac{R(t_0)}{x}},$$

where (using (2.13) and (2.6))

$$R'(t_0) = \frac{A_1}{2\sqrt{t_0}},\quad x = A_1\sqrt{t_0} + a_0(t-t_0)\frac{2a_0\sqrt{t_0}}{A},$$

whence

$$\frac{R(t_0)}{x} = \frac{1}{1+\frac{2a_0^2 t}{A_1^2}}$$

and

$$\frac{P}{P_\Phi(0)} = \frac{A_1^2/2a_0^2 t}{1+A_1^2/2a_0^2 t}. \tag{2.19}$$

Along the $0y$ axis near B (Figure 10)

$$\frac{P}{P_\Phi(0)} = -\frac{\partial}{\partial y}\int_0^{r_0^*}\frac{r\,dr}{\sqrt{r^2+y^2}}$$

according to (2.2), where

$$r_0^* = A_1 \sqrt{t - \frac{1}{a_0}\sqrt{r_0^{*2} + y^2}}, \quad r_0^{*2} = \frac{A_1^2\left(t - \frac{y}{a_0}\right)}{1 + A_1^2/2a_0^2\, t}, \quad t - \frac{y}{a_0} \to 0.$$

After performing the calculation this gives relation (2.19) at the point $y = a_0 t$.

Thus, both in the two-dimensional and axisymmetric problems in the case $R'(0) = \infty$, the point B is nonsingular for the linear solution. The pressure, determined by formulas (2.10) and (2.12), gives a satisfactory description of the phenomenon.

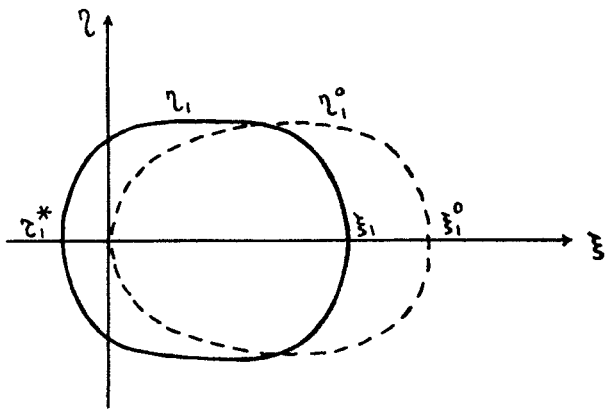


FIGURE 9.

When $P_\phi(t)$ is constant, the pressure decreases monotonically from A to B. For $P_\phi(t)$, which decreases sharply with time, the pressure rises towards the point B; this is shown for particular $P_\phi(t)$, $R(t)$, $f\left(\frac{x}{R\varepsilon}\right)$ /27/ in Figure 11.

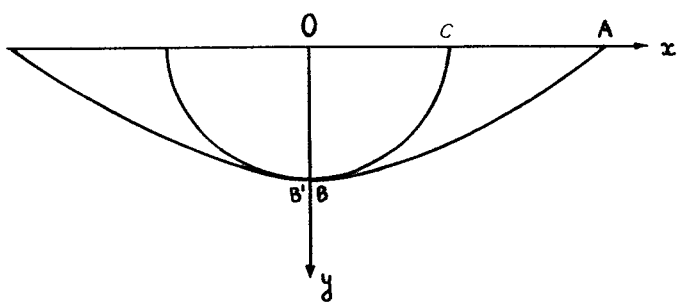


FIGURE 10.

If the pressure at the surface decreases sharply with time, the disturbed region, where the pressure is nonzero, has the form of a fairly small strip near the region BB' of the initial values. The whole perturbed motion is concentrated near the shock front. The energy carried by the wave for finite pressures is obviously small in this case, the pressure variation along the ordinate being given in Figure 12.

For pressures falling sharply with time but assuming particular values at the surface, there are a number of simple formulas to illustrate the above statement /27/. In addition, when the pressures vary slightly asymptotic formulas exist for $V \gg a_0$ and $V \ll a_0$ /27/.

§ 3. Neighborhood of the sonic line

It is of interest to study the vicinity of the sonic line to determine whether a shock wave forms there (cf. § 3, Chapter I).

According to § 3 in Chapter I a shock wave forms in the second approximation, if in the linear approximation the pressure near the sonic line $r_1 = a_0 t$ has the singularity $\sqrt{\left|1 - \frac{r_1}{a_0 t}\right|}$, and in (3.18) $A < 0$ behind the line (i.e., for $r_1 < a_0 t$) and $B \leqslant 0$ in front of the line (i.e., for $r_1 > a_0 t$). The solution is now found for arbitrary $R'(t)$ and, in particular $R'(0) = \infty$.

Consider the two-dimensional case. First the behavior of the solution behind $r_1 = a_0 t$ is studied, where for simplicity we initially set

$$\frac{P_1}{\sqrt{(\xi - x)^2 + y^2 + \eta^2}} = 1$$

in the integrand of (2.7), i.e., we consider the variation of the area in Figure 9.

Expression (2.7) then yields

$$P = -\frac{1}{\pi}\frac{\partial}{\partial y}\int_0^{\xi_0} d\xi \int_0^{\eta_1^0} d\eta - \frac{1}{\pi}\frac{\partial}{\partial y}\int_{r_1^*}^{0} d\xi \int_0^{\eta_1'} d\eta - \frac{1}{\pi}\frac{\partial}{\partial y}\int_0^{\xi_1} d\xi \int_0^{\eta_1} d\eta + \frac{1}{\pi}\frac{\partial}{\partial y}\int_0^{\xi_0} d\xi \int_0^{\eta_1^0} d\eta, \quad (3.1)$$

where the first integral on the right-hand side gives the pressure on the sonic line, and ξ_0, η_1^0 equal ξ_1, η_1 on $x^2 + y^2 = a_0^2 t^2$; the form (3.1) was chosen for convenience.

Consider the second integral in (3.1). From (2.8)

$$r_1^* = \frac{a_0^2 t^2 - x^2 - y^2}{2f'(0)\, a_0^2 t + 2x},$$

$$\frac{\partial \eta_1^0}{\partial y} = \frac{-y}{\sqrt{2f'(0)\,\xi^2 a_0^2 + 2f'(0)\,\xi t a_0^2 - (\xi - x)^2 + a_0^2 t^2 - y^2}};$$

hence

$$-\frac{1}{\pi}\frac{\partial}{\partial y}\int_{r_1^*}^{0} d\xi \int_0^{\eta_1} d\eta = 2y\sqrt{\frac{a_0^2 t^2 - x^2 - y^2}{f'(0)\, t\, a_0^2 + x}}. \quad (3.2)$$

The third and fourth integrals in (3.1) are now considered subject to the condition

$$\frac{P_1}{\sqrt{(\xi - x)^2 + y^2 + \eta^2}} = 1.$$

It is clear that

$$-\frac{1}{\pi}\frac{\partial}{\partial y}\int_0^{\xi_1} d\xi \int_0^{\eta_1} d\eta + \frac{1}{\pi}\frac{\partial}{\partial y}\int_0^{\xi_0} d\xi \int_0^{\eta_1^0} d\eta =$$

$$= \frac{1}{\pi}\int_{\xi_0}^{\xi_1} \frac{y\,d\xi}{\sqrt{(f(\xi)\,a_0 - a_0 t)^2 - (\xi - x)^2 - y^2}} +$$

$$+ \frac{1}{\pi}\int_0^{\xi_0} \frac{y\,d\xi}{\sqrt{(f(\xi)\,a_0 - a_0 t)^2 - (\xi - x)^2 - y^2}} -$$

$$- \frac{1}{\pi}\int_0^{\xi_0} \frac{y\,d\xi}{\sqrt{f^2(\xi)\,a_0^2 - 2f(\xi)\,t a_0^2 - \xi^2 + 2\xi x}}. \qquad (3.3)$$

The radicand in the first integral near the sound wave can be written as

$$(f - t)^2 a_0^2 - (\xi - x)^2 - y^2 = a_1(\xi - \xi_0) + a_0^2 t^2 - x^2 - y^2,$$
$$a_1 = 2(f(\xi_0) - t)\,a_0^2 f'(\xi_0) - 2\xi_0 + 2x.$$

In (2.2) we may put

$$a_0^2 (f - t)^2 - \xi^2 + 2\xi x - a_0^2 t^2 = -\xi(\xi - \xi_0)\,\varphi(\xi), \qquad (3.4)$$
$$\varphi(\xi_0) \neq 0,\ \varphi(0) \neq 0.$$

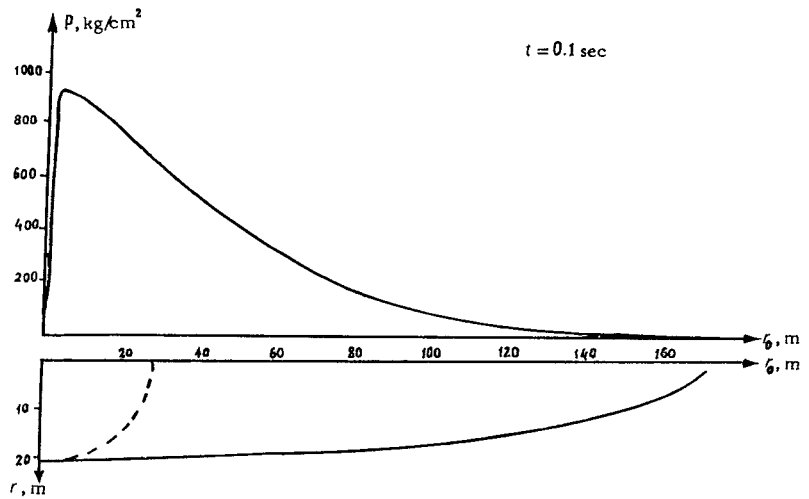


FIGURE 11.

In virtue of (2.8), $\xi = 0$ and $\xi = \xi_0$ are single roots on the left-hand side of (3.4); the first integral of (3.3) is therefore

$$-\frac{y}{f(\xi_0)\, a_0^2 f'(\xi_0) - a_0^2 t f'(\xi_0) - \xi_0 - x} \sqrt{a_0^2 t^2 - x^2 - y^2}\,. \tag{3.5}$$

The difference between the second and third integrals gives leading-order small quantities near the points $\xi = 0$ and $\xi = \xi_0$, at which the integrand becomes singular.

Using this and dividing the integral into two parts, namely, 0 to $\frac{\xi_0}{2}$ and $\frac{\xi_0}{2}$ to ξ_0, we can write

$$\frac{1}{\pi}\int_0^{\xi_0} \frac{y\,d\xi}{\sqrt{-\xi(\xi-\xi_0)\,\varphi(\xi) + a_0^2 t^2 - x^2 - y^2}} -$$

$$-\frac{1}{\pi}\int_0^{\xi_0} \frac{y\,d\xi}{\sqrt{-\xi(\xi-\xi_0)\,\varphi(\xi)}} = \frac{y}{\varphi(\xi_0)} \frac{\sqrt{a_0^2 t^2 - x^2 - y^2}}{\xi_0/2}\,. \tag{3.6}$$

Consider a pressure at the surface in the form $P_1 = P_\Phi(t)$. If the values of the function $\frac{P_\Phi(t')}{\sqrt{(\xi-x)^2 + \eta^2 + y^2}}$ at the points 0 and ξ_0 are taken into account, we finally obtain

$$P = -\frac{1}{2\pi}\frac{\partial}{\partial y}\int_0^{\xi_0} d\xi \int_{-\eta_1^0}^{\eta_1^0} \frac{P_\Phi(t')\,d\eta}{\sqrt{(\xi-x)^2 + y^2 + \eta^2}} +$$

$$+\frac{1}{\pi}\frac{y\sqrt{a_0^2 t^2 - x^2 - y^2}}{f'(0)\, t a_0^2 + x}\frac{P_\Phi(0)}{a_0 t} -$$

$$-\frac{1}{\pi}\frac{P_\Phi(0)}{\frac{\varphi(0)\,\xi_0}{2}}\frac{y\sqrt{a_0^2 t^2 - x^2 - y^2}}{a_0 t} - \frac{1}{\pi}\frac{y\sqrt{a_0^2 t^2 - x^2 - y^2}}{\varphi(\xi_0)\,\xi_0/2} \times$$

$$\times \frac{P_\Phi(f(\xi_0))}{a_0 t - a_0 f(\xi_0)} + \frac{1}{\pi} P_\Phi(f(\xi_0)) \frac{y\sqrt{a_0^2 t^2 - x^2 - y^2}}{\xi_0 - x - (f-t)\, a_0^2 f'(\xi_0)} \times$$

$$\times \frac{1}{a_0 t - a_0 f(\xi_0)} \tag{3.7}$$

for $x^2 + y^2 \leqslant a_0^2 t^2$, where

$$\varphi(0) = \frac{2x - 2a_0^2 f'(0)\, t}{\xi_0},\quad \varphi(\xi_0) = \frac{2\xi_0 - 2x - 2a_0^2 f'(\xi_0)(f(\xi_0) - t)}{\xi_0}.$$

At points satisfying $x^2 + y^2 \geqslant a_0^2 t^2$

$$P = -\frac{1}{2\pi}\frac{\partial}{\partial y}\int_{\xi_2}^{\xi_1} d\xi \int_{-\eta_1}^{\eta_1} \frac{P_\Phi(t')\,d\eta}{\sqrt{(\xi-x)^2 + y^2 + \eta^2}}\,, \tag{3.8}$$

where $\xi_{1,2}$ are the two roots of equation (2.8).

On $x^2 + y^2 = a_0^2 t^2$ the roots are ξ_1^0, $\xi_2^0 = 0$. If we use (2.6) and divide the integral with respect to ξ into two parts between the limits ξ_2 to $\frac{\xi_1}{2}$ and $\frac{\xi_1}{2}$ to ξ_1, and in addition set

$$\frac{P_\Phi}{\sqrt{(\xi - x)^2 + y^2 + \eta^2}} = 1, \quad \xi_1^0 = \xi^0,$$

then

$$\frac{1}{\pi} y \int_{\xi_2}^{\xi_1} \frac{d\xi}{\sqrt{-\xi(\xi - \xi^0)\varphi(\xi) + a_0^2 t^2 - x^2 - y^2}}$$

$$- \frac{1}{\pi} y \int_0^{\xi^0} \frac{d\xi}{\sqrt{-\xi(\xi - \xi^0)\varphi(\xi)}} = \frac{1}{\pi} y \int_{\xi_2}^{\frac{\xi_1}{2}} + \frac{1}{\pi} y \int_{\frac{\xi_1}{2}}^{\xi_1} -$$

$$- \frac{1}{\pi} y \int_0^{\frac{\xi_2}{2}} - \frac{1}{\pi} y \int_{\frac{\xi_2}{2}}^{\xi_2}.$$

The following notation is introduced:

$$-\xi(\xi - \xi^0)\varphi + a_0^2 t^2 - x^2 - y^2 = -(\xi - \xi_2)(\xi - \xi_1)\varphi_1(\xi).$$

The derivative of the first integral with respect to ξ_2 is now calculated, where

$$\lambda = \int_{\xi_2}^{\xi_1/2} \frac{d\xi}{\sqrt{(\xi - \xi_2)(\xi_1 - \xi)\varphi_1(\xi)}}.$$

The change of variables $\xi - \xi_2 = \zeta$ yields

$$\frac{d\lambda}{d\xi_2} = \frac{d}{d\xi_2} \int_0^{\frac{\xi_1}{2} - \xi_2} \frac{d\zeta}{\sqrt{\zeta(\xi_1 - \xi_2 - \zeta)\varphi_1(\xi_2 + \zeta)}} =$$

$$= \frac{\frac{1}{2}\frac{d\xi_1}{d\xi_2} - 1}{\sqrt{\left(\frac{\xi_1}{2} - \xi_2\right)\frac{\xi_1}{2}\varphi_1\left(\frac{\xi_1}{2}\right)}} + \int_0^{\frac{\xi_1}{2} - \xi_2} \frac{d\zeta}{-2\sqrt{\zeta\varphi_1}(\xi_1 - \zeta - \xi_2)^{3/2}}.$$

Since, in virtue of (2.8),

$$\xi_2 = \frac{x^2 + y^2 - a_0^2 t^2}{2x}, \quad \xi_1 - \xi^0 = \frac{a_0^2 t^2 - x^2 - y^2}{2\xi^0 - 2x - 2f'(f - t)a_0^2} \tag{3.9}$$

near $r_1 = a_0 t$, $\frac{dt}{d\xi_2}$ is finite when $r_1 = a_0 t$. In a similar way the finiteness of the derivatives of all the remaining integrals with respect to $(r_1 - a_0 t)$ can be shown. The expression for the pressure in front of the sonic line, i. e., for $r_1 > a_0 t$, will then contain terms to the first power of $r_1 - a_0 t$.

Since there are no singularities of the square-root type, it can be shown (by applying the method of § 3 in Chapter I) that the pressure in front of the sound wave differs from the pressure on $r_1 = a_0 t$ to an order higher than second. In the case $R'(0) = \infty$, (3.7) shows that the pressure behind the sonic line also agrees with the pressure on the sonic line itself, expressed by the integral in (3.7), since the terms containing $\sqrt{a_0^2 t^2 - x^2 - y^2}$ on the right-hand side of (3.7) cancel.

Thus, for $P_1(x, t) = P_\Phi(t)$ in the case $R'(0) = \infty$ the pressure distribution both in front and behind the sonic line does not contain terms with either $\sqrt{1 - \frac{r_1}{a_0 t}}$ for $r_1 < a_0 t$ or $\sqrt{\frac{r_1}{a_0 t} - 1}$ for $r_1 > a_0 t$, and therefore a shock wave does not form in front of BC (Figure 10).

This fact is in good agreement with the absence of a singularity at the intersection B of AB, BC and the $0y$ axis (Figure 10).

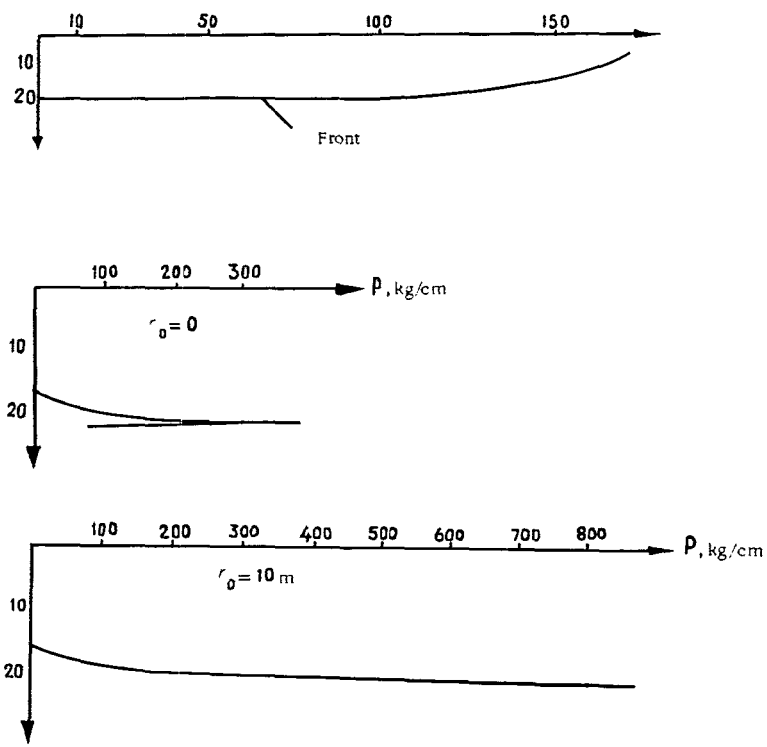


FIGURE 12.

When $P_1(x, t) = P_\Phi(t) f\left(\frac{x}{R}\right)$ (Figure 12), the pressure distribution will have the same singularity as in the case $P_\Phi(t) = P_1^0 = \text{const}$. The investigation is then identical to that in § 3 of Chapter I with no shock wave in the second approximation. This conclusion does not refer to higher approximations; for $P_\Phi(t) \ll P_\Phi(0)$ a shock wave forms in front of BC, but to an order higher than second. Thus, when solving nonstationary problems, one must show in the second approximation inside the region the absence of a shock wave in front of the sonic line, or in its presence calculate its magnitude (as in § 3, Chapter I).

§ 4. Shock wave AB in the nonlinear case

In § 2 we considered the linear problem of determining the pressure on the shock wave AB. It is of interest, however, to obtain a more exact value for the pressure on the shock wave. In addition, there are instances when the linear theory is inapplicable (see §§ 6 and 7).

A method of obtaining an approximate solution is now discussed. It is assumed that the compatability relation along the characteristics of the first family is satisfied up to the shock wave (/11/, Volume 4) and one may substitute in it the conditions on the shock wave.

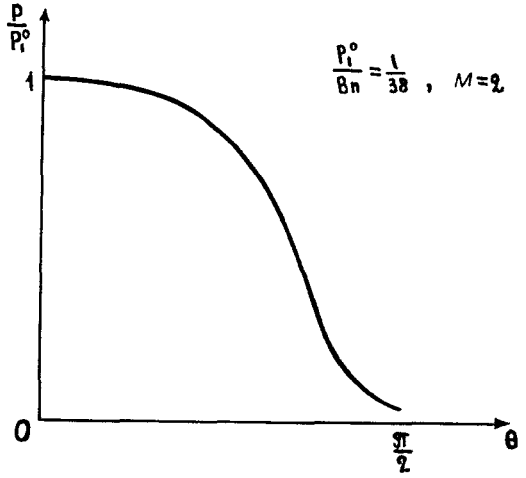


FIGURE 12a.

The improved solution obtained in the first approximation agrees with the solution obtained by the technique of replacing the linear characteristics by more exact ones. It is only in the determination of the pressure in the second approximation that the arguments are nonrigorous, characteristic of the technique of substituting shock relations in the compatibility condition.

Consider the two-dimensional problem, from which the axisymmetric one can be derived. The equations of motion are taken in the form (F) for

Cartesian coordinates. Condition (C) is satisfied at the fluid boundary. A variation in the fluid surface does not affect the shock wave to second order.

If it is assumed that $R'(t) > a_0$ and that the shock wave is so weak that it does not affect the flow behind it, the shock wave AB (Figure 10) is found to move in the fluid. We will consider the interaction of rarefaction waves moving from the boundary with the shock front. The rarefaction waves are taken as the characteristic surfaces of the first family, which overtake the shock front. The equation of the characteristics of the first family, $y(x,t)$, can be found by the usual method /35/.

The operators $\left(\frac{\partial}{\partial x}\right)$ and $\left(\frac{\partial}{\partial y}\right)$ denote differentiation along a characteristic surface, where

$$\left(\frac{\partial P}{\partial x}\right) = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial t}\frac{\frac{\partial y}{\partial x}}{\frac{\partial y}{\partial t}}; \quad \left(\frac{\partial P}{\partial y}\right) = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial t}\frac{1}{\frac{\partial y}{\partial t}}. \tag{4.1}$$

The velocity components u and v along the axes (x and y) are derived from equations (4.1) by replacing P by u and v.

Substituting (4.1) in the equations of motion (F), and bearing in mind that the solution is nonunique, the equation for the characteristic is found to be given by /35/

$$\frac{\partial y}{\partial x} = -\operatorname{tg}\alpha, \quad \frac{\partial y}{\partial t}\cos\alpha = a + v_1, \; u = v_1 \cdot \sin\beta, \tag{4.2}$$

where $y(x, t)$ is the equation of the characteristic and v_1 is the velocity of the fluid particles; the particle velocities in (4.2) were assumed normal to the characteristic, which is true for waves overtaking a front ($\alpha \approx \beta$, $\cos(\beta - \alpha) = 1$).

The compatibility condition for equations (F) on a characteristic has the form

$$(a\sin\alpha + u)\left(\frac{\partial P}{\partial x}\right) + (a\cos\alpha + v)\left(\frac{\partial P}{\partial y}\right) + \rho a(u\sin\alpha + a)\left(\frac{\partial u}{\partial x}\right) +$$

$$+ \rho a u\cos\alpha\left(\frac{\partial u}{\partial y}\right) + \rho a v\sin\alpha\left(\frac{\partial v}{\partial x}\right) + \rho a(v\cos\alpha + a)\left(\frac{\partial v}{\partial y}\right) = 0. \tag{4.3}$$

For pressures up to 1,000 kg/cm^2, $\frac{P}{Bn}$ is small, e.g., for water $\frac{P}{Bn} = 0.05$.

The solution of (4.2) and (4.3) is sought in the form

$$P = P_1 + \frac{1}{Bn}P_2, \; y = y_1 + \frac{1}{Bn}y_2, \; \alpha = \alpha_1 + \frac{1}{Bn}\alpha_2,$$

$$a = a_0\left(1 + \frac{n-1}{2Bn}P_1\right), \; dV = \frac{dP}{\rho a}, \; V^2 = u^2 + v^2, \tag{4.4}$$

the last two equations in (4.4) following from (A) and the conditions on the shock front. If (4.4) is substituted in (4.2) and (4.3) and only first-order small quantities are retained, the following equations result:

$$\frac{\partial y_1}{\partial t}\cos\alpha_1 = a_0,\quad \frac{2}{a_0^2}\frac{dP}{dt} + \frac{P_1}{a_0}\frac{d\alpha_1}{ds} = 0, \tag{4.5}$$

where $\frac{d}{dt} = \left(\frac{\partial}{\partial x}\right) a_0 \sin\alpha_1 + \left(\frac{\partial}{\partial y}\right) a_0 \cos\alpha_1$ and $\frac{d\alpha_1}{ds}$ is the curvature of the characteristic.

The solution of (4.5) for the linear problem finally yields

$$x - R(t_0) = a_0 (t - t_0)\sin\alpha_1(t_0), \qquad y_1 = a_0 (t - t_0)\cos\alpha_1(t_0), \tag{4.6}$$

$$P = P_1(R, t_0)\frac{\sqrt{\frac{r(t_0)}{a_0}}}{\sqrt{t - t_0 + \frac{r(t_0)}{a_0}}} = \frac{A_0(t_0)}{\sqrt{t - t_0 + \frac{r(t_0)}{a_0}}}, \tag{4.7}$$

where, at time t_0, $R(t_0) = x$ is the coordinate of a point E of the boundary when the wave front passes through it (§ 2) and $r(t_0)$ is the radius of curvature of the front.

For the solution corresponding not to the front AB itself (Figure 12), but to an arbitrary characteristic $A'B'$, the boundary condition for (4.5) is taken at the point $x = \lambda R(t')$ at time $t' = t_0 + t_0'$, where $\lambda \ll 1$. Then, in place of (4.6) and (4.7), the equation of the curve $A'B'$ (Figure 12) is easily obtained in the form

$$x - \lambda R(t_0' + t_0) = a_0 (t - t_0 - t_0')\sin\alpha_1(t_0 + t_0'), \qquad y_1 = a_0 (t - t_0 - t_0')\cos\alpha_1(t_0 + t_0') \tag{4.8}$$

in addition to the solution on the characteristics

$$P_1 = \frac{A_0(t_0 + t_0')}{\sqrt{t - t_0' - t_0 + \frac{r(t_0 + t_0')}{a_0}}}, \tag{4.9}$$

where

$$A_0(t') = P_1(\lambda R, t')\sqrt{\frac{r(t')}{a_0}} \tag{4.10}$$

is constant.

One should equate second-order terms in $\frac{P_1}{Bn}$ in (4.2) to determine y_2:

$$\frac{\partial y_2}{\partial t} + a_0 \sin\alpha_1(t_0)\frac{\partial y_2}{dx} = a_0 \frac{n+1}{2}\frac{P_1}{\cos\alpha_1(t_0)}. \tag{4.11}$$

Equation (4.11) can be expressed in terms of the variables t_0, t with the aid of the first equation of (4.6). It can be easily shown that

$$\left.\frac{\partial y_2}{\partial t}\right|_{t_0} = \left.\frac{\partial y_2}{\partial t}\right|_{x} + \left.\frac{\partial y_2}{\partial x}\right|_{t} a_0 \sin \alpha_1 (t_0),$$

and then from (4.11) we obtain

$$\left.\frac{\partial y_2}{\partial t}\right|_{t_0} = a_0 \frac{n+1}{2} \frac{A_0 (t_0)}{\sqrt{t - t_0 + \frac{r (t_0)}{a_0}}} \frac{1}{\cos \alpha_1 (t_0)}.$$

The solution of this equation is

$$y_2 = a_0 \frac{n+1}{2} \frac{A_0 (t_0)}{\cos \alpha_1 (t_0)} 2 \sqrt{t - t_0 - \frac{r (t_0)}{a_0}} - a_0 \frac{n+1}{2} \frac{A_0 (t_0)}{\cos \alpha_1 (t_0)} 2 \sqrt{t' - t_0 - \frac{r (t_0)}{a_0}}, \tag{4.12}$$

where the relation

$$\lambda R (t') - R (t_0) = a_0 (t' - t_0) \sin \alpha_1 (t_0)$$

can be used to determine t' from the boundary condition and (4.8).

Equation (4.12) indicates that the motion of the characteristics will be defined along linear rays (cf. § 2).

In our approximation $t' = t_0$. To determine the motion of the shock wave, we write the formula for its velocity ((2.2), Chapter I) as follows:

$$\frac{\frac{\partial y}{\partial t}}{\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}} = a_0 \left(1 + \frac{n+1}{2} \frac{P_1}{Bn}\right). \tag{4.13}$$

Consider (4.8) and (4.6). In virtue of the smallness of $\frac{P}{Bn}$, characteristics overtaking the front remain a distance of the order of $\frac{P_1}{Bn}$ ahead of it. Therefore, t_0' and $\lambda' = 1 - \lambda$ are small.

Expanding (4.8) in powers of these parameters and using (4.6), we have

$$t_0' = \frac{\lambda' R (t_0)}{- a_0 \sin \alpha_1 (t_0) + a_0 (t - t_0) \cos \alpha_1 (t_0) \frac{\partial \alpha_1 (t_0)}{\partial t_0} + R' (t_0)}, \tag{4.14}$$

$$y_1 = a_0 (t - t_0) \cos \alpha_1 (t_0) - \lambda' R (t_0) \frac{a_0 (t - t_0) \sin \alpha_1 (t_0) \frac{\partial \alpha_1 (t_0)}{\partial t_0} + a_0 \cos \alpha_1 (t_0)}{R' (t_0) + a_0 (t - t_0) \cos \alpha_1 (t_0) \frac{\partial \alpha_1 (t_0)}{\partial t_0} - a_0 \sin \alpha_1 (t_0)}. \tag{4.15}$$

Now $y = y_1 + \frac{1}{Bn} y_2$, and if (4.15) and (4.12) are substituted in (4.13) and only first-order small quantities retained, an equation is derived for λ' along the front, namely

$$-\frac{\partial}{\partial t}\left(R(t_0) B_0(t, t_0) \lambda'\right)\Big|_{t_0} = -a_0 \frac{n+1}{4} \frac{P_1}{Bn} \frac{1}{\cos \alpha_1},$$

$$B_0(t, t_0) = \frac{a_0 (t - t_0) \sin \alpha_1 \frac{\partial \alpha_1}{\partial t_0} + a_0 \cos \alpha_1}{R'(t_0) + a_0 (t - t_0) \cos \alpha_1 \frac{\partial \alpha_1}{\partial t_0} - a_0 \sin \alpha_1}$$

The solution of this equation subject to the boundary condition $\lambda' = 0$, $x = R(t)$ is

$$R(t_0) \lambda' = a_0 \frac{n+1}{4} \frac{1}{\cos \alpha_1 (t_0)} \frac{A_0(t_0)}{Bn} 2\left(\sqrt{t - t_0 + \frac{r_E}{a_0}} - \sqrt{\frac{r_E}{a_0}}\right) B_0(t, t_0). \quad (4.16)$$

If we substitute λ' from (4.16) in (4.15), the shock-front equation is found to be

$$y = y_1 + \frac{1}{Bn} y_2$$

or

$$y = a_0 (t - t_0) \cos \alpha_1 (t_0) + \frac{1}{2Bn} y_2, \quad (4.17)$$

where y_2 is given by (4.12). The pressure can be found from formula (4.9), where in the first approximation $t' = t_0$, $\lambda' = 0$.

The solution (4.17), (4.9) is linear, taken along the more exact characteristics. Further, the first approximation can be improved by substituting in (4.3) the conditions on the shock front and collecting second-order small terms:

$$2\frac{dP_2}{dt} + \frac{P_2}{t - t_0 + \frac{r(t_0)}{a_0}} + \psi(x, t) = 0,$$

where ψ is determined in terms of P_1 /35/. Calculations by these formulas are cumbersome and yield a fairly small improvement in the first approximation.

For the axisymmetric problem, equation (4.12) of the improved characteristics becomes

$$y = y_0 + \frac{1}{Bn} y_2;$$

y_0 is the linear solution of (2.6), and

$$y_2 = \frac{a_0}{\cos \alpha_1 (t_0)} \frac{n+1}{2} P_1 \frac{\sqrt{\frac{r(t_0)}{a_0}} \sqrt{R(t_0)}}{\sqrt{a_0 \sin \alpha_1 (t_0)}} 2 \ln \frac{\sqrt{t - t_1} + \sqrt{t - t_2}}{\sqrt{\frac{r(t_0)}{a_0}} + \sqrt{\frac{R(t_0)}{a_0 \sin \alpha_1 (t_0)}}},$$

where

$$t_1 = t_0 - \frac{r(t_0)}{a_0},\ t_3 = t_0 - \frac{R(t_0)}{a_0 \sin \alpha_1(t_0)}.$$

At the point B (Figure 8), where the sphere $r_1 = a_0 t$ is tangent to the envelope (2.6) of the disturbances, we have

$$x = \frac{a_0^2 t}{R'(0)},\ t_0 = 0,\ y = a_0 t \sqrt{1 - \frac{a_0^2}{R'^2(0)}}$$

in the linear case. Substituting these values in the formula $y = y_0 + \frac{1}{2Bn} y_2$ for the shock wave, a relation is obtained for t_0' near the point B. For the case $R'(0) = \infty$, however, (2.6) indicates that $x = 0$, $t_0' = 0$ at B, and y for the shock wave will be determined by its improved value. In the two-dimensional case at the point B

$$y_2 = \frac{n+1}{2a_0} P_\Phi(0) A_1^2 \left(\sqrt{t + \frac{A_1^2}{2a_0^2}} - \sqrt{\frac{A_1^2}{2a_1^2}}\right),\ R(t) = A_1 \sqrt{t}.$$

§5. Pressure on AB by the method of characteristics

Consider the second approximation in the two-dimensional problem. If equations (F) are written in Cartesian coordinates x, y and we set

$$v_x = v_{x_1} + v_{x_2},\ v_y = v_{y_1} + v_{y_2},$$
$$P = P_1 + P_2,$$
$$\rho = \rho_0 + \rho_0 \frac{P_1}{Bn},$$

where the subscript 1 refers to the first approximation and the subscript 2 to the second, and if we further introduce another small parameter for the independent variable $t = u + t_1(x, y)$, then equations (F) take the form

$$\frac{\partial v_{x_2}}{\partial u} + \frac{1}{\rho_0}\frac{\partial P_2}{\partial x} = F_1,\ \frac{\partial v_{y_2}}{\partial u} + \frac{1}{\rho_0}\frac{\partial P_2}{\partial y} = F_2,\ \frac{\partial P_2}{\partial u} + \rho_0 a_0^2 \left(\frac{\partial v_{x_2}}{\partial x} + \frac{\partial v_{y_2}}{\partial y}\right) = F_3, \qquad (5.1)$$

$$F_1 = -v_{x_1}\frac{\partial v_{x_1}}{\partial x} - v_{y_1}\frac{\partial v_{x_1}}{\partial y} + \frac{1}{\rho_0}\frac{P_1}{Bn}\frac{\partial P_1}{\partial x} + \frac{1}{\rho_0}\frac{\partial P_1}{\partial u}\frac{\partial t_1}{\partial x},$$
$$F_2 = -v_{x_1}\frac{\partial v_{y_1}}{\partial x} - v_{y_1}\frac{\partial v_{y_1}}{\partial y} + \frac{1}{\rho_0}\frac{P_1}{Bn}\frac{\partial P_1}{\partial y} + \frac{1}{\rho_0}\frac{\partial P_1}{\partial u}\frac{\partial t_1}{\partial y},$$
$$F_3 = -v_{x_1}\frac{\partial P_1}{\partial x} - v_{y_1}\frac{\partial P_1}{\partial y} - \rho_0 a_0^2 n \frac{P_1}{Bn}\left(\frac{\partial v_{x_1}}{\partial x} + \frac{\partial v_{y_1}}{\partial y}\right) +$$
$$+ \rho_0 a_0^2 \left(\frac{\partial v_{x_1}}{\partial u}\frac{\partial t_1}{\partial x} + \frac{\partial v_{y_1}}{\partial u}\frac{\partial t_1}{\partial y}\right) \qquad (5.2)$$

to second order.

To determine the pressure P_2 on the shock wave AB (Figure 5), which is done in § 4 of Chapter I by means of the potential φ, we use the characteristics of equations (5.1). Along $u = u(x, y)$ we have

$$\left(\frac{\partial P}{\partial x}\right) = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial u}\frac{\partial u}{\partial x}$$

with similar relations for v_x and v_y along the characteristic $u = u(x, y)$. Substituting these relations in (5.2) and writing the compatability condition for (5.1) along $u = u(x, y)$, we obtain the equation

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \frac{1}{a_0^2}$$

for $u(x, y)$, and the relation

$$-\rho_0 a_0^2\left(\frac{\partial v_{x_2}}{\partial x}\right) - \rho_0 a_0^2\left(\frac{\partial v_{y_2}}{\partial y}\right) - a_0^2\frac{\partial u}{\partial x}\left(\frac{\partial P_2}{\partial x}\right) - a_0^2\frac{\partial u}{\partial y}\left(\frac{\partial P_2}{\partial y}\right) + \\ + F_3 + \rho_0 a_0^2\frac{\partial u}{\partial x}F_1 + \rho_0 a_0^2\frac{\partial u}{\partial y}F_2 = 0, \qquad (5.3)$$

involving P_2, v_{x_2} and v_{y_2}.

In order to find a relation between v_{x_2}, v_{y_2} and P_2, we express v_x, v_y, P in terms of the potential φ according to (4.39) in Chapter I:

$$v_x = \frac{\partial \varphi_1}{\partial x}\Big|_t + \frac{\partial \varphi_2}{\partial x}\Big|_t;\ v_{x_2} = \frac{\partial \varphi_2}{\partial x}\Big|_u - \frac{\partial \varphi_1}{\partial u}\frac{\partial t_1}{\partial x},\ v_{y_2} = \frac{\partial \varphi_2}{\partial y}\Big|_u - \frac{\partial \varphi_1}{\partial u}\frac{\partial t_1}{\partial y},$$
$$v_{x_1} = -f_1\frac{1}{\rho_0}\frac{\partial u}{\partial x}H(u-\tau_1) + \frac{f_2}{\rho_0}\frac{\partial u}{\partial x}(u-\tau_1) + \frac{1}{\rho_0}\frac{\partial f_1}{\partial x}(u-\tau_1),$$
$$v_{y_1} = -f_1\frac{1}{\rho_0}\frac{\partial u}{\partial y}H(u-\tau_1) + \frac{f_2}{\rho_0}\frac{\partial u}{\partial y}(u-\tau_1) + \frac{1}{\rho_0}\frac{\partial f_1}{\partial y}(u-\tau_1), \qquad (5.4)$$

and the pressure is found from the Lagrange representation. If, on the shock wave corresponding to the characteristic $u = \tau_1(x, y)$ of the linear problem (see § 4, Chapter I), $t_1(x, y)$ is taken as the improvement to the linear equation of the shock wave then $P_2 = -\rho_0\frac{\partial \varphi_2}{\partial u}$. Hence, (5.4) yields

$$v_{x_2} = \frac{1}{\rho_0}\frac{\partial \tau_1}{\partial x}P_2 + \frac{P_1}{\rho_0}\frac{\partial t_1}{\partial x},\ v_{y_2} = \frac{1}{\rho_0}\frac{d\tau_1}{dy}P_2 + \frac{P_1}{\rho_0}\frac{\partial t_1}{\partial y} \qquad (5.5)$$

to second order on the shock wave.

The obtained relations can also be derived from the formula $v = \frac{P}{\rho_0 D}$ and from the formula for the shock wave velocity to first order

$$D = a_0 - a_0^2\frac{\partial \tau_1}{\partial x}\frac{\partial t_1}{\partial x} - a_0^2\frac{\partial \tau_1}{\partial y}\frac{\partial t_1}{\partial y}.$$

If $v_x = v\cdot\cos\beta$, $v_y = v\cdot\sin\beta$, then

$$\operatorname{tg}\beta = \frac{\frac{\partial \tau_1}{\partial y} + \frac{\partial t_1}{\partial y}}{\frac{\partial \tau_1}{\partial x} + \frac{\partial t_1}{\partial x}},\ \beta = \beta_0 + \beta_1,\ \frac{\beta_1}{\cos^2\beta_0} = \frac{\frac{\partial \tau_1}{\partial y}}{\frac{\partial \tau_1}{\partial x}} + \frac{\frac{\partial t_1}{\partial y}}{\frac{\partial \tau_1}{\partial x}} - \frac{\frac{\partial t_1}{\partial x}\frac{\partial \tau_1}{\partial y}}{\left(\frac{\partial \tau_1}{\partial x}\right)^2}.$$

The expressions for F_1, F_2, F_3 in (5.2) are calculated from (4.39) in Chapter I. When (5.5) is substituted in (5.4), a relation is obtained for P along AB. If terms containing the factor $H(u-\tau_1)$ are retained, where $H(x)$ is a unit function, (5.3) and (5.4) yield

$$2\left(\frac{\partial P_2}{\partial x}\right)\frac{\partial \tau_1}{\partial x} - 2\left(\frac{\partial P_2}{\partial y}\right)\frac{\partial \tau_1}{\partial y} - P_2\left(\frac{\partial^2 \tau_1}{\partial x^2} + \frac{\partial^2 \tau_1}{\partial y^2}\right) + \frac{\partial f_1}{\partial x}\frac{\partial t_1}{\partial x} +$$
$$+ \frac{\partial f_1}{\partial y}\frac{\partial t_1}{\partial y} + f_1\left(\frac{\partial^2 t_1}{\partial x^2} + \frac{\partial^2 t_1}{\partial y^2}\right) - \frac{n+1}{2a_0^2} f_1 f_2 - f_1 \frac{\partial f_1}{\partial x}\frac{\partial \tau_1}{\partial x} -$$
$$- f_1 \frac{\partial f_1}{\partial y}\frac{\partial \tau_1}{\partial y} = 0. \qquad (5.6)$$

Expression (5.6) does not differ from the corresponding expression of §4 in Chapter I, obtained there from equation (4.36) by a method using rays. The method mentioned here is more commonly used. Terms in (5.3) containing $\delta(u-\tau_1)$ vanish after performing the quadratures of (5.6), in virtue of (4.44) in Chapter I. Equation (5.6) can also be obtained with the aid of rays, not as applied to equation (4.36) in Chapter I, but to system (5.1). If we set

$$v_{x_2} = v^0_{x_2} + v^1_{x_2}(u-\tau_1)$$

near the front $u = \tau_1(x, y)$ and introduce a similar notation for the other variables, then (5.1) yields the following:

$$v^1_{x_2} - \frac{1}{\rho_0}\frac{\partial \tau_1}{\partial x} P^1_2 = -v^0_{x_2}\delta(u-\tau_1) + \frac{1}{\rho_0} P^0_2 \delta(u-\tau_1)\frac{\partial \tau_1}{\partial x} - \frac{1}{\rho_0}\frac{\partial P^0_2}{\partial x} + F_1,$$

$$v^1_{y_2} - \frac{1}{\rho_0}\frac{\partial \tau_1}{\partial y} P^1_2 = -v^0_{y_2}\delta(u-\tau_1) + \frac{1}{\rho_0} P^0_2 \delta(u-\tau_1)\frac{\partial \tau_1}{\partial y} - \frac{1}{\rho_0}\frac{\partial P^0_2}{\partial y} + F_2,$$

$$\frac{1}{\rho_0 a_0} P^1_2 - a_0\frac{\partial \tau_1}{\partial x} v^1_{x_2} - a_0\frac{\partial \tau_1}{\partial y} v^1_{y_2} = -\frac{P^0_2}{\rho_0 a_0}\delta(u-\tau_1) +$$

$$+ a_0\frac{\partial \tau_1}{\partial x} v^0_{x_2}\delta(u-\tau_1) + a_0\frac{\partial \tau_1}{\partial y} v^0_{y_2}\delta(u-\tau_1) - a\frac{\partial v^0_{x_2}}{\partial x} - a_0\frac{\partial v^0_{y_2}}{\partial y} + F_3. \qquad (5.7)$$

Relations (5.5) are derived by equating terms with $\delta(u-\tau_1)$. The determinant of the resulting system vanishes, and therefore the condition for the existence of a solution to the system requires that the right-hand side of (5.7) be orthogonal to the eigenvector $\left(\frac{\partial \tau_1}{\partial x}, \frac{\partial \tau_1}{\partial y}, \frac{1}{a_0}\right)$ of (5.7). Condition (5.6) is again obtained when $P_2 = P_2^0$; if (5.6) is integrated along the characteristic $u = \tau_1$, terms containing $\delta(u-\tau_1)$ vanish in virtue of (4.44) in Chapter I, where $t_1(x, y)$ is taken as the improvement to the linear equation $u = \tau_1$.

The solution at large times is found in the next section, but some preliminary observations are in order here. According to (2.4) in Chapter II,

$$P = -\frac{1}{\pi}\frac{1}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}\int_0^{\psi_1} d\psi \int_{r_2^*}^{r_1^*} P_\Phi(t') f\left(\frac{r}{R(t')}\right) r dr$$

for large t, and in the case of finite $R(\infty)$ it can be shown that $\frac{\partial P}{\partial r_1} \gg \frac{\partial P}{r_1 \partial \theta}$, i.e., the motion takes place along linear rays, and $r^*_{0,2}, \psi_1$ are constant on the characteristics $\zeta = \text{const}, \zeta = t - \frac{r_1}{a_0}$. Formula (5.2) may be rewritten in the form

$$P = \frac{F(\zeta)}{a_0 t}, \tag{5.7}$$

where

$$F(\zeta) = \frac{1}{\pi}\frac{\partial}{a_0 \partial \zeta}\int_0^{\psi_1} d\psi \int_{r_0^*(\zeta,\psi)}^{r_2^*(\zeta,\psi)} P_\Phi(t') f\left(\frac{r}{R(t')}\right) r dr. \tag{5.8}$$

The equation of the characteristics is (/27/, p. 170)

$$s = a_0(t-\zeta) + F(\zeta)\frac{n+1}{2}\frac{1}{\rho_0 a_0^3}\ln\frac{t}{\zeta} + R(t_0).$$

When the equation of the characteristics is substituted in formula (2.2) in Chapter I for the shock velocity, we find that along the front

$$\frac{F^2(\zeta)}{\rho_0 a_0}\ln\frac{t}{\zeta} = \frac{4}{n+1}\int_{\zeta_0}^{\zeta} F(\zeta)\, d\zeta, \tag{5.9}$$

where one can assume that the upper limit $\zeta = \zeta_0$ corresponds to $F(\zeta_0) = 0$. When $R(\infty)$ is infinite (5.9) may not be valid, and other formulas accounting for the departure from one-dimensionality can be used along AB. Relation (5.7) holds near the $0y$ axis, and along the $0y$ axis

$$F(\zeta) = \frac{1}{2a_0}\frac{\partial}{\partial \xi} P_\Phi(\zeta) R^2(\zeta).$$

The most interesting case is $R(t) = Vt$ as $t \to \infty$, $V \ll a_0$. For $P_\Phi \sim \frac{1}{t}$ we have $F(y_1) \sim V^2$, i.e., $F = \text{const}$; for $P_\Phi = \frac{1}{(1+t)^2}$, $F(y_1) \sim \frac{V^2}{y_1^2}$ and from (5.9) $y_1 \sim \sqrt{V\sqrt{\ln y_1}}$ where, in virtue of $V \ll a_0$, the motion is one-dimensional along the radius. When the motion is not one-dimensional everywhere then, for large t, one-dimensional solutions along r near the Oy axis should be marked with one-dimensional rays CM (/27/ Figure 8, §4, Chapter 6) of a two-dimensional nonstationary short wave. The same holds near the surface /39/.

§6. Motion of the shock wave for large times

The treatment of §§2 and 4 referred to finite times. When $R'(0) < \infty$, the solution near BB' (Figure 8) will no longer be determined for large times by the initial values of P_Φ and R' alone. The solutions (3.6) and (5.21) of Chapter I are therefore invalid. For large times, disturbances propagating over a compressed fluid overtake the shock wave and, although they are small, a large time interval produces qualitatively a new attenuation effect.

This fact is known for the one-dimensional case /13, 14, 4/ as well as for short waves /18/, and will be used to determine the shock front BB' (Figure 8). The solution will also be valid for $R'(0) = \infty$, since the only requirement is the finiteness of $t - \frac{r_1}{a_0}/\ln r_1$.

Thus, for large times one can obtain the pressure on the shock wave at finite distances from the axis of symmetry.

The method is based on replacing the linear characteristics by improved ones /4/, so enabling the pressure to be found on the shock wave. The solution is also shown to satisfy the equations of gas dynamics.

Consider the axisymmetric case, and assume that the pressure at the surface decreases with time and vanishes at time t_0. If t_0 is small, the whole motion is concentrated in the vicinity of the front /18/; for finite t_0, we have an N-wave /5/. Alternatively, it can be assumed that as $t \to \infty$, $P_\Phi(t) \to 0$.

First consider the pressure on the $0y$ axis. According to §2, in the linear case

$$P = \frac{1}{a_0}\frac{1}{2y}\frac{\partial}{\partial t'} P_\Phi(t')\, R^2(t'),$$

$$t' = t - \frac{y}{a_0},\quad P_1(x, t) = P_\Phi(t) \tag{6.1}$$

along $0y$ for large y or t.

The linear characteristic variable $t - \frac{y}{a_0}$ is replaced in (6.1) by the improved one, y_1. Retaining first-order small quantities with respect to $\frac{1}{y}$ in (3.2) in Chapter I and integrating the one-dimensional equation for the characteristics $y_1 = \text{const}$, we have

$$t = \frac{y}{a_0} - \frac{n+1}{2}\frac{1}{a_0 B n}\frac{\partial}{\partial y_1}\frac{P_\Phi(y_1)\, R^2(y_1)}{2a_0}\ln\frac{y}{a_0 y_1} + y_1. \tag{6.2}$$

The following notation is used:

$$\beta(y) = \frac{1}{a_0 B n}\frac{n+1}{2}\ln\frac{y}{a_0 y_1}, \tag{6.3}$$

$$F(y_1) = \frac{\partial}{\partial y_1}\frac{P_\Phi(y_1)\, R^2(y_1)}{2a_0} \tag{6.4}$$

If (6.2) is substituted in Pfreim's formula for the shock velocity, the following relation is derived on the shock wave /4/:

$$\beta(y) = \frac{2\int_0^{y_1} F(y')\,dy'}{F^2(y_1)}$$

or

$$F(y_1) = \sqrt{\frac{2a_0\int_0^{y_0} F(y')\,dy'}{\frac{n+1}{2}\frac{1}{Bn}\ln\frac{y}{a_0 y_1}}}, \tag{6.5}$$

where y_0 is taken as the upper limit of the integral, since when $y \to \infty$, $F(y_1) \to 0$ and $y_1 \to y_0$, y_0 being the root of F. Suppose $P_\Phi(t_0) = 0$ or $P_\Phi(t) \to 0$ as $t \to \infty$, while $R(\infty)$ remains finite. According to Rolle's theorem, $F(y_0)$ vanishes for $0 < y_0 < t_0$. Since, in virtue of (6.5), $F(y_1) \to 0$ as $y \to \infty$, we conclude that y_1 may be replaced by y_0 in the radicand.

The shock-front equation is found from (6.2) and (6.5) in the form

$$t = \frac{y}{a_0} - \sqrt{\beta(y)}\sqrt{2\int_0^{y_0} F(y')\,dy'} + y_0.$$

The pressure at the front is

$$P = \frac{F(y_1)}{y}. \tag{6.6}$$

The extreme characteristic is found from (6.2) and is

$$t = \frac{y}{a_0} + y_0.$$

The condition for the formation of a tail shock is found from the envelope of the characteristics (6.2) (Figure 13):

$$t = \frac{y}{a_0} - \beta(y)F(y_1) + y_1,$$
$$\beta(y)F'(y_1) = 1.$$

A tail shock forms if $F'(y_1) > 0$.

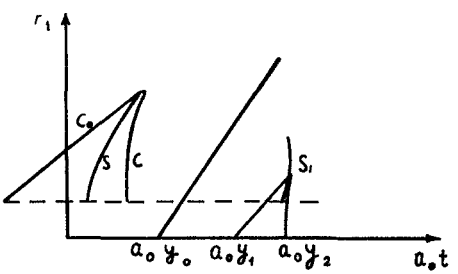


FIGURE 13.

TABLE 1. Pressure on the shock fronts as $t \to \infty$

y_1'	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{3}{5}$	$\frac{3}{4}$	1	$\frac{3}{2}$	2	4	10
$\chi(y_1')$	$\frac{1}{4}$	$\frac{7}{30}$	$\frac{7}{90}$	$\frac{1}{15}$	$\frac{1}{16}$	$\frac{1}{21}$	$\frac{1}{48}$	$\frac{1}{225}$	$\frac{1}{228}$	$\frac{1}{100}$	$\frac{1}{240}$
$\frac{\chi(y_1')-\frac{2Pa_1^2}{P_1a_1^2}}{t'}$				$\frac{1}{60}$	$\frac{1}{160}$	$\frac{1}{302}$	0				$-\frac{3}{10^4}$
t'	0	1	2	4	10	42	∞				20

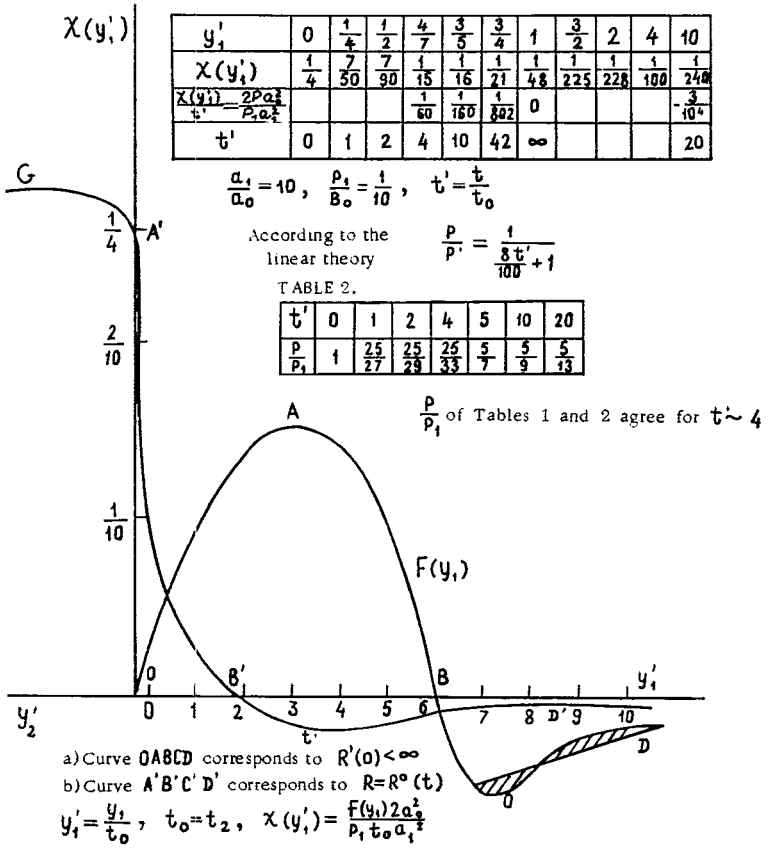
$\frac{a_1}{a_0} = 10$, $\frac{P_1}{B_0} = \frac{1}{10}$, $t' = \frac{t}{t_0}$

According to the linear theory $\frac{P}{P'} = \frac{1}{\frac{8t'}{100}+1}$

TABLE 2.

t'	0	1	2	4	5	10	20
$\frac{P}{P_1}$	1	$\frac{25}{27}$	$\frac{25}{29}$	$\frac{25}{33}$	$\frac{5}{7}$	$\frac{5}{9}$	$\frac{5}{13}$

$\frac{P}{P_1}$ of Tables 1 and 2 agree for $t' \sim 4$



a) Curve OABCD corresponds to $R'(0) < \infty$

b) Curve A'B'C'D' corresponds to $R = R^0(t)$

$y_1' = \frac{y_1}{t_0}$, $t_0 = t_2$, $\chi(y_1') = \frac{F(y_1)2a_0^2}{P_1 t_0 a_1^2}$

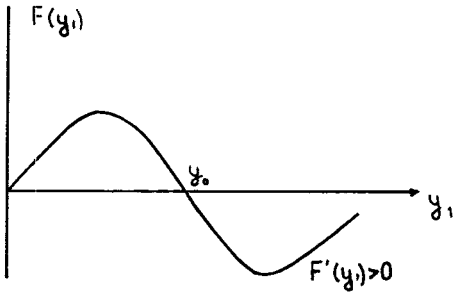


FIGURE 14, 14'.

It can be easily shown that for given boundary values $P_\phi\ (y_1)$ and $R\ (y_1)$ the condition $F'\ (y_1) > 0$ is satisfied (Figure 14), and a tail shock forms in which the pressure rises to its unperturbed value. For a boundary pressure of the form

$$P_\phi\ (t)\ f\left(\frac{x}{R}\right)$$

the previous solution is slightly modified and multiplied by

$$2\int_0^1 \xi f\ (\xi)\ d\xi.$$

Now consider the whole region BB' on the shock wave. In the linear case the pressure is given by formula (2.2). For large $r_1 = \sqrt{x^2 + y^2}$,

$$P = -\frac{1}{2\pi}\frac{1}{r_1}\frac{\partial}{\partial y}\int_0^{2\pi} d\psi \int_0^{r_0^*} r dr\ P_1\left(r, t - \frac{1}{a_0} r_1 + \frac{1}{a_0} r_1 \cos\theta \cos\psi\right) \tag{6.7}$$

in polar coordinates, i. e., the same effect as for a concentrated force.

The solution (6.7) can be rewritten as

$$P = -\frac{1}{2\pi}\frac{\sin\theta}{a_0}\frac{1}{r_1}\frac{\partial}{\partial t'}\int_0^{2\pi} d\psi \int_0^{r_0^*} r dr\ P_1\left(r, t' + \frac{1}{a_0} r_1 \cos\theta \cos\psi\right), \tag{6.8}$$

where, from the approximation for large r_1,

$$\frac{\partial}{\partial y} = \sin\theta\ \frac{\partial}{\partial t'}.$$

The characteristics of the linear problem are now replaced by the improved ones, where $t - \frac{r_1}{a_0} = y_1$. Expression (6.8) then becomes

$$P = \frac{F\ (y_1)}{r_1}, \tag{6.9}$$

where

$$F\ (y_1) = \frac{1}{2\pi}\frac{\sin\theta}{a_0}\frac{\partial}{\partial y_1}\int_0^{2\pi} d\psi \int_0^{r_0^*} P_1\left(r, y_1 + \frac{1}{a_0} r_1 \cos\theta \cos\psi\right) r dr,$$

$$r_0^* = R\left(y_1 + \frac{1}{a_0} r_0^* \cos\theta \cos\psi\right) \tag{6.10}$$

and the equation of the characteristics are

$$t = \frac{r_1}{a_0} - \frac{1}{a_0 B n}\frac{n+1}{2}\ F\ (y_1) \ln \frac{r_1}{a_0 y_1} + y_1. \tag{6.11}$$

The equation of the characteristics is derived from relation (3.2) of Chapter I:

$$1 - v_{r_1}\frac{\partial t}{\partial r_1} - v_\theta \frac{\partial t}{r_1 \partial \theta} = \sqrt{\left(\frac{\partial t}{\partial r_1}\right)^2 + \left(\frac{\partial t}{r_1 \partial \theta}\right)^2}\; a_0\left(1 + \frac{n+1}{2}\frac{P}{Bn}\right), \quad (6.12)$$

where v_{r_1} and v_θ are the radial and transverse components of the velocity vector. It follows from (6.9) that $v_{r_1} = 0\left(\frac{1}{r_1}\right)$, $v_\theta = 0\left(\frac{1}{r_1^2}\right)$. Then, using the fact that $t = \frac{r_1}{a_0} + 0(\ln r_1)$ and retaining in (6.12) small quantities of the order of $\frac{1}{r_1}$, we find that the derivatives with respect to θ may be ignored and, since on the shock wave $V_{r_1} \approx a_0 \frac{P}{Bn}$, the equation of the characteristics is given by (6.11). Pfreim's relation on the shock front gives /4/

$$\frac{1}{a_0 Bn}\frac{n+1}{2}\ln\frac{r_1}{a_0 y_1} = \frac{2\int_0^{y_1} F(y_1)\,dy_1}{F^2(y_1)} \quad (6.13)$$

or

$$F(y_1) = \sqrt{\frac{\frac{\sin\theta}{\pi}\int_0^{2\pi} d\psi \int_0^{r_0^*(y_0)} P_1\left(r, y_0 + \frac{1}{a_0} r\cos\theta\cos\psi\right) r\,dr}{\frac{1}{Bn}\frac{n+1}{2}\ln\frac{r_1}{a_0 y_0}}}; \quad (6.14)$$

as before, for boundary pressures of the given type (Figure 14, 14'), y_0 is found from the relation $F(y_0) = 0$ where, according to Rolle's theorem, $0 < y_0 < t_0$.

The equation of the shock front has the form

$$t = \frac{r_1}{a_0} - \frac{n+1}{2}\frac{F(y_1)}{a_0 Bn}\ln\frac{r_1}{a_0 y_1} + y_1, \quad (6.15)$$

where $F(y_1)$ is given by (6.14).

The velocity potential is

$$\varphi = -\frac{1}{\rho_0 r_1}\int_0^{y_1} F(y)\,dy, \quad (6.16)$$

a more exact solution being indicated in /26/.

The condition for the tangential shock velocity is

$$a_0\frac{\partial \varphi}{\partial r_1} + \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial t}\frac{\partial f}{\partial r_1} = 0, \quad (6.17)$$

where $f = r_1 - a_0 t$ is determined from formula (6.15), equation (6.17) being satisfied to first order.

The tangential shock velocity is

$$v_s = \frac{\partial \varphi}{\partial r_1} \frac{\partial t}{r_1 \partial \theta} + \frac{\partial \varphi}{r_1 \partial \theta} \sim \frac{1}{r_1^{\frac{3}{2}} \sqrt{\ln r_1}} . \tag{6.18}$$

In the two-dimensional case, in place of (6.7) we obtain from §2 for large r_1

$$P = \frac{F(y_1)}{\sqrt{r_1}},$$

$$F(y_1) = \frac{\sqrt{2}}{2\pi} \sin\theta \frac{\partial}{\partial y_1} \int_0^{\sqrt{y_1 a_0}} d\eta_1 \int_{-r_1^*}^{r_0^*} P_1 \left(|\xi|,\ y_1 + \frac{\xi}{a_0} \cos\theta - \frac{\eta_1^2}{a_0} \right) d\xi, \tag{6.19}$$

$$r_0^* = R\left(y_1 + \frac{1}{a_0} r_0^* \cos\theta \right),$$

$$r_1^* = R\left(y_1 - \frac{\cos\theta}{a_0} r_1^* \right), \tag{6.20}$$

and the equation of the characteristics in the form

$$t = \frac{r_1}{a_0} - \frac{n+1}{2} \frac{1}{a_0 B n} 2\sqrt{r_1}\ F(y_1) + y_1. \tag{6.21}$$

The pressure on the shock front is obtained in the same way as (6.13), and is valid not only for $r_1 < a_0 t$ but also for $r_1 > a_0 t$. In formula (6.7) one should take $-\psi_1 < \psi < \psi_1$, $r_2^* < r < r_1^*$ as integration limits, as in formula (2.4). The form of the function $F(y_1)$ corresponds to the linear solution for the pressure; $y_1 = 0$ corresponds to $B_1 C_1$ (Figure 5), $y_1 < 0$ is the transition to the region AB_1C_1, the solution is continued up to the shock wave AB_1, where $y = y_2$, and F abruptly vanishes ($A'G$ in Figure 14', where $y_1 < 0$). In the integral appearing in the radicand of formula (6.14) the integration limits should be $y_2 < y_1 < y_0$, and in the region AB_1C_1, where $y_2 < y_1 < 0$, $F(y_1)$ should take the form of (2.4). Apparently the rarefaction will always occur behind $r_1 = a_0 t$ (/27/, p. 41 and Figure 12) and $y_0 > 0$. When $x = 0$, $y_2 = 0$ and the section $A'G$ is missing. The solution is also valid for $R'(0) = \infty$, since we did not require the pressure to vanish on BB' in the linear case; the only condition for which the solution holds is that y_1 or $\frac{t - \frac{r_1}{a_0}}{\ln r_1}$ remains finite. The same is also true for the two-dimensional problem.

The solutions (6.9), (6.11) and (6.19), (6.21) are now shown to satisfy the equations of motion. These solutions correspond to one-dimensional short waves, for which $\frac{\partial}{\partial t'} \gg \frac{\partial}{\partial r_1}\Big|_{t_1} \gg \frac{\partial}{\partial \theta}$, and therefore the nonstationary equations of short wave /2/ are used.

In terms of the variables $r_1 = a_0 t \left(1 + \frac{n+1}{2} M_0 \delta \right)$, $\frac{P}{Bn} = M_0 \mu = \frac{V_{r_1}}{a_0}$, $\tau = \ln t$, $M_0 = \frac{P_\Phi(0)}{Bn}$ the equation governing the two-dimensional case is derived

from (F):

$$\frac{\partial \mu}{\partial \tau} - (\mu - \delta)\frac{\partial \mu}{\partial \delta} + \frac{1}{2}\mu = 0.$$

Its characteristics are given by

$$C_1 = \ln t^{1/4}\mu, \quad C_2 = \frac{\delta - 2\mu}{\mu^2},$$

and the general solution is given in terms of the function Φ:

$$\delta = 2\mu + \frac{1}{t}\Phi(t^{1/4}\mu). \tag{6.22}$$

New notation is introduced in (6.19) and (6.21) yielding

$$M_0\delta = \frac{F(y_1)}{Bn}\frac{2}{\sqrt{t}} - \frac{2}{n+1}\frac{y_1}{t}$$

or

$$M_0\delta = 2\frac{P}{Bn} - \frac{2}{n+1}\frac{y_1}{t}.$$

According to (6.19) $y_1 = f(P\sqrt{t})$, where f is the inverse function of F. Hence

$$\delta = 2\mu - \frac{2}{n+1}\frac{1}{M_0}\frac{1}{t}f(P\sqrt{t}),$$

or, with the notation

$$-\frac{2}{n+1}\frac{1}{M_0}f(P\sqrt{t}) = \Phi(t^{1/4}\mu),$$

we find that equation (6.22) is satisfied.

We have demonstrated the validity of solution (6.19), (6.21), obtained by replacing the linear characteristics by improved ones. Incidentally, (6.22) gives $\mu t^{1/4} = \Phi^{-1}(\delta t - 2\mu t)$ or $\mu t^{1/4} = \Phi^{-1}\left(r_1 - a_0 t - 2\frac{P}{Bn}\frac{n+1}{2}\right)$ where, for finite t, Φ^{-1} is found by comparison with (6.19) and $\Phi^{-1} \equiv F$. The axisymmetric problem obeys

$$\frac{\partial \mu}{\partial \tau} + (\mu - \delta)\frac{\partial \mu}{\partial \delta} + \mu = 0.$$

The characteristics of this equation are

$$C_1 = \ln t\mu, \quad C_2 = \frac{\delta}{\mu} + \ln \mu,$$

and the general solution /2/ is

$$\delta = -\mu \ln \mu + \frac{t_0}{t}\Phi(t\mu), \tag{6.23}$$

where t_0 is constant.

The solution (6.9), (6.11) is now expressed in terms of the variables δ, μ, where

$$M_0\delta = \frac{1}{a_0 t}\frac{F(y_1)}{Bn}\ln\frac{r_1}{a_0 y_1} - \frac{2}{n+1}\frac{y_1}{t},$$

$$\delta = \frac{P}{Bn}\frac{1}{M_0}\ln\left(\frac{r_1}{a_0 F}\frac{F}{y_1}\right) - \frac{2}{n+1}\frac{y_1}{M_0 t}.$$

Since $P = \frac{F(y_1)}{t}$ and $r_1 \approx a_0 t$, then $y_1 = \Phi_1(Pt)$ and

$$\delta = -\mu\ln\mu + \mu\ln\frac{F(y_1)}{y_1 M_0 Bn} - \frac{2}{n+1}\frac{1}{M_0}\frac{\Phi_1(Pt)}{t}.$$

However $P = \frac{F(y_1)}{t}$, $y_1 = \Phi_1(Pt)$, $t\mu = \frac{F(y_1)M_0}{M_0 Bna_0}$, and if we also set

$$\frac{F(y_1)}{M_0 Bna_0}\ln\frac{F(y_1)}{y_1 M_0 Bn} - \frac{2}{n+1}\frac{1}{M_0}\Phi_1(Pt) = t_0\Phi(t\mu),$$

(6.23) is derived.

Hence, shock fronts are determined by the Whitham technique /5/ and then checked by the short-wave method /2/. The assumption that $R(\infty) < \infty$ is unnecessary. In particular, if $R^2(t) = A^2 t$, $P_\Phi = \frac{1}{1+t}$ then $F(y_1) = \frac{A^2}{2(1+y_1)^2}$, and from (6.5) $y_0 \sim \ln^{\frac{1}{4}}\frac{y}{a_0 y_0}$. For $R(t) = Vt$, $y_0 \sim \ln\frac{y}{a_0 y_0}$, $F(y_1) = V^2$.

§7. Transition of the pressure front surface velocity through the velocity of sound

It was pointed out earlier (§1) that in a real problem the velocity $R'(t)$ is initially infinite and at some time $t = \tilde{t}$ equals the velocity of sound in the fluid. The pressure front then moves over the surface with a subsonic velocity, and the shock front breaks away from the section at which the pressure is applied and emerges to the free surface of the fluid.

The problem is first treated in a linear formulation. Suppose, according to §1, $D(t_2) < a_0$; (1.1) then indicates that at some instant $t = \tilde{t}$, $R'(t) = a_0$.

The perturbed fluid motion at time $t < \tilde{t}$ is delineated by AB (Figure 8), whose equation, from (2.6), is

$$x = R(t_0) + a_0(t - t_0)\cos\theta,$$
$$y = a_0(t - t_0)\sin\theta, \qquad (7.1)$$

where $\cos\theta = \frac{a_0}{R'(t_0)}$; (7.1) is also the equation of the ray MC (Figure 8), θ being the angle between the ray and $0x$.

It is obvious that $t_0 = t$ at the point $x = R(t), y = 0$. When $t = \bar{t}$ we have $R'(t_0) = a_0$ at the point $x = R(\bar{t})$, since $R'(\bar{t}) = a_0$.

At time $t = \bar{t}$ the shock front arrives at the point $\theta = 0$ on the fluid boundary with zero pressure; the shock front is then perpendicular to the surface.

Further, the front velocity in the linear case is equal to a_0, and the velocity of the section of the pressure front at the surface is $R'(t) < a$ for $t > \bar{t}$. The shock front therefore breaks away from the section to which the pressure is applied and emerges to the free surface.

Figure 1 shows that the angle of incidence of the air wave is given by

$$\sin\lambda = \frac{D(t_0 + t)}{R'(t)},$$

and at time $t = \bar{t}$ in the linear case

$$R'(\bar{t}) = a_0, \quad \sin\lambda = \frac{D(t_0 + \bar{t})}{a_0}.$$

From Snell's Law the angle of refraction is given by

$$\frac{\sin\lambda}{\sin\theta} = \frac{a_0}{D(t_0 + t)}$$

and $\theta = \frac{\pi}{2}$ at time $t = \bar{t}$. This case thus corresponds to total internal reflection, and it is obvious that in the nonlinear case too the same will occur.

To determine the next stage in the motion of the front, we note that in the linear case each point of the front moves along a ray. The inclination of the rays MC (Figure 8) to $0x$ becomes smaller and smaller as $y \to 0$, and at the point A when $t = \bar{t}$ the ray $\theta = 0$ coincides with $0x$. Further, the motion of the front $ABB'A'$ (Figure 5) takes place along the same rays, the coordinates of the front being found from (7.1), where already $t_0 \leqslant \bar{t}$. The coordinate of A is

$$x = R(\bar{t}) + a_0(t - \bar{t}).$$

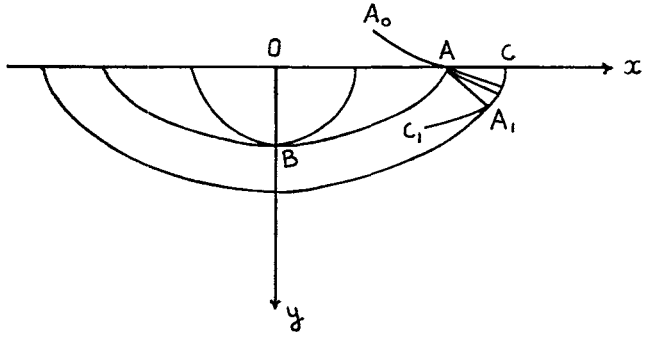


FIGURE 15.

The pressure (given by (2.12)) at the front is obtained by the ray representation for the axisymmetric case in the form

$$P_M = \frac{\sqrt{-\frac{R'^3(t_0)}{R''(t_0)}\frac{\sin^2\theta}{a_0^2}}\sqrt{\frac{R(t_0)}{x}}}{\sqrt{t-t_0-\frac{R'^3(t_0)}{R''(t_0)}\frac{\sin^2\theta}{a_0^2}}} P_1(x_0, t_0), \tag{7.2}$$

where $x_0 = R(t_0)$, $\sin^2\theta = 1 - \frac{a_0^2}{R'^2(t_0)}$. At the point C (Figure 15) the pressure $P_M = 0$, since $\theta = 0$. At A for time $t = \bar{t}$ the radius of curvature of the front is

$$r_A = -\frac{R'^3(t_0)\sin^2\theta}{R''(t_0)\,a_0}$$

and since $\theta = 0$, $r_A = 0$. The point A then lies on the caustic A_0A /23/, the evolute of the front AB:

$$\xi = R(t_0) + \frac{R'^3(t_0)}{R''(t_0)}\frac{\sin^2\theta\cos\theta}{a_0^2},$$
$$\eta = \frac{R'^3(t_0)\sin^3\theta}{R''(t_0)\,a_0^2}. \tag{7.3}$$

where for $\sin\theta = 0$, $\eta = 0$ and $\xi = R(\bar{t})$.

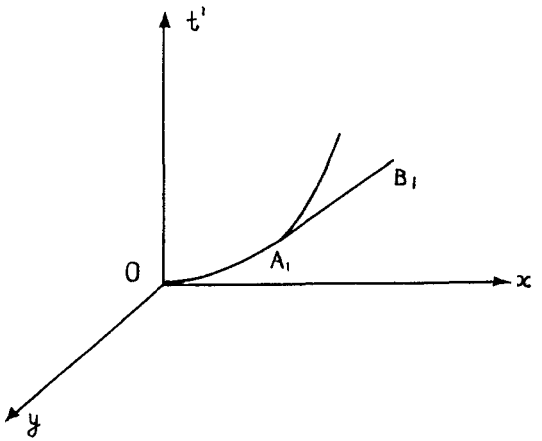


FIGURE 16.

Now consider the region of integration for solution (2.4) in the space (t', x', y'), where $x' = r\cos\psi$, $y' = r\sin\psi$ (Figure 16).

When $t' = t$ the surface $r = R(t')$ is inclined at $t'R'(\bar{t}) = a_0$ to the t' axis at the point A (Figure 16). The equation of the surface A_1B_1 is

$$r = R(\bar{t}) + a_0(t' - \bar{t}). \tag{7.4}$$

Consider a hyperboloid, passing through the point $r = x$, $t' = t$ on the surface (7.4) and given by

$$t' = t - \frac{1}{a_0}\sqrt{x^2 + r^2 - 2xr\cos\varphi + y^2},$$

which on A_1B_1 ($y=0$) transforms into an asymptotic cone; this cone intersects $r = R(t')$ at $t' = \tilde{t}$.

On the front CA_1 it follows from the above that $t' \leqslant \tilde{t}$. Thus, the pressure is given by (2.4) on the entire front CA_1 and is nonzero everywhere with the exception of C.

If $\tilde{t} \ll t$, the front CA_1 tends to a sphere. In this case, the time during which $R'(t)$ is greater than a_0 is small and the pressure during the time $t > \tilde{t}$ differs considerably from the subsonic case $R'(0) < a_0$. Thus, a small initial period of supersonic velocities subsequently influences the pressure distribution, which will no longer necessarily be a second-order quantity on the front CA_1, as in the case of subsonic velocities, but according to (1.1) and (2.12) will be of order $P \sim P_\Phi \frac{D}{a_0}\frac{(\tilde{t}+t_2)^1}{tt_2^2}$ where, for the case under consideration and $\tilde{t} \ll t_2$, $\frac{D}{a_0}$ will be small for finite t. Thus, the order of smallness of P depends on $\frac{D}{a_0}$.

The same problem is now considered for the nonlinear formulation. Irregular reflection occurs near the fluid surface for $t \gg \tilde{t}$, and the linear theory does not hold.

An attempt is made to improve the linear theory by the method of § 4. The phenomenon taking place is analyzed qualitatively with a view to formulating the exact problem and determining the approximate solution for the shock wave.

For the improvement to the linear theory formulas (4.7) and (4.17) are written on the shock wave in the form

$$P = P_\Phi(t_0)\frac{\sqrt{\frac{r(t_0)}{a_0}}}{\sqrt{t - t_0 + \frac{r(t_0)}{a_0}}}, \tag{7.5}$$

where the equation of the shock front CA_1 (Figure 15) is

$$x = R(t_0) + a_0(t - t_0)\sin\alpha_1(t_0),$$
$$y = a_0(t - t_0)\cos\alpha_1(t_0) +$$
$$+ a_0\frac{n+1}{2}\frac{P_\Phi(t_0)}{En}\frac{\sqrt{\frac{r(t_0)}{a_0}}}{\cos\alpha_1(t_0)}\left(\sqrt{t - t_0 + \frac{r_E}{a_0}} - \sqrt{\frac{r_E}{a_0}}\right), \tag{7.6}$$

and the radius of curvature is

$$r_E = r(t_0),\quad \sin\alpha_1(t_0) = \frac{a_0}{R'(t_0)} = \cos\theta,$$
$$\frac{r(t_0)}{a_0} = -\frac{R'^3(t_0)}{a_0^2 R''(t_0)}\sin^2\theta. \tag{7.7}$$

When the velocity of the front over the surface $R'(t) > a_0$, the front (7.6) moves along the linear rays, $t_0 = \text{const}$, $t_0 \leqslant \tilde{t}$ and the pressure on each ray is given by (7.5). If at some moment $\tilde{t}$ the velocity $R'(t)$ equals the velocity of the shock front $a_0\left(1+\frac{n+1}{4}\frac{P}{Bn}\right)$ we have the situation shown in Figure 15. At time $t=\tilde{t}$ the front AB (Figure 15) breaks away from the boundary region of the pressure (since the angle between the front and the fluid surface $0x$ becomes equal to $\frac{\pi}{2}$) and emerges to the free surface.

The pressure in the linear case is given by (7.5), where $t_0 \leqslant \tilde{t}'$ and the value of $\tilde{t}'$ corresponds to the instant at which $R'(t)=R'(\tilde{t}')=a_0$. As pointed out, there is a focusing of rays at the point A resulting from the linear theory; it can be eliminated /10/ by replacing the linear characteristics by improved ones. According to § 4 this is carried out by replacing the parameter t_0 by $t' = t_0 + t_0'$ in the solution of the linear problem, where t_0' is found from (4.14).

At time $t = \tilde{t}'$ at the point A, (4.14) yields

$$t_0' = \frac{\frac{n+1}{4}a_0\frac{P_\Phi}{Bn}\,(\sqrt{3}-\sqrt{2})}{\frac{3}{2}R''}$$

and from (7.7) the radius of curvature of the front at A is $r_A(t_0) = -2t_0'a_0$.

Thus, in the exact theory when the shock front emerges to the fluid surface at time $t = \tilde{t}'$ its curvature is not infinite and the point A does not lie on the caustic.

The linear solution therefore ceases to be valid near the fluid surface and there is irregular reflection from the surface. This phenomenon is investigated qualitatively.

At the point A (7.5) yields that for $t \ll \tilde{t}'$, $P = P_\Phi(t)$ and for $t = \tilde{t}'$, $t_0 = \tilde{t}'$, $P = \sqrt{\frac{2}{3}}P_\Phi(\tilde{t}')$. Further, for different values of $t > \tilde{t}'$ the pressure varies from first to second order in $\frac{P_\Phi}{Bn}$. At time $t = \tilde{t}$, $R'(\tilde{t})$ equals the shock-front velocity along the normal, and so the front is inclined at an angle $\frac{\pi}{2}$ to the surface. Subsequently the front moves over the zero-pressure surface. The treatment of /6/ can therefore be applied to the short wave arising near the surface (irregular reflection), taking the curved front into consideration, in order to obtain a qualitative picture of the phenomenon and some approximate formulas.

The equation of the unperturbed shock front AB (Figure 15) is given by (7.6). Waves of reduced pressure, which arise at time $t \geqslant \tilde{t}$ at points on the free surface, slow down the motion of the front in some region near the point A. The equations of these waves are the characteristic equations of short waves /20/, and the limiting line A_1C_1, which separates the zone of surface interference, is the characteristic passing through the point $x = R(\tilde{t})$ on the surface at time $t = \tilde{t}$.

For a qualitative investigation and in order to obtain simple working formulas, disturbances moving from the surface are treated as Riemann waves. The boundary of influence of these waves is obtained by equating the velocity of Riemann waves at pressure P to the shock velocity component along the radius vector at the point on the shock wave (the origin of the polar system is chosen at the point $x = R(t), y=0$). The angle λ between the normal to the shock front and the radius vector AA_1 (Figure 15) is then

$$\lambda = \sqrt{\frac{n+1}{2}\frac{P}{Bn}}, \tag{7.8}$$

where P is given by (7.5). The polar angle θ at A_1 is the sum of λ and the angle at which the normal is inclined to the front $0x$, namely $\left(-\frac{\partial x}{\partial y}\right)$.

Since $\theta = \frac{y}{x - R(\tilde{t})}$, we have

$$\frac{y}{x - R(\tilde{t})} = \sqrt{\frac{n+1}{2}\frac{P}{Bn}} + \left(-\frac{\partial x}{\partial y}\right). \tag{7.9}$$

With the aid of (7.6) and (7.7) relation (7.9) yields an equation for t_0 or θ at the point A_1 on the shock wave, which bounds the zone of action on the surface. If (7.7) is substituted in (7.9), we find

$$\begin{aligned}(t_1 - t_1')\sqrt{\frac{2R''t_1'}{a_0}} + \frac{\frac{n+1}{2}}{t_1}\frac{P_\Phi}{Bn}\sqrt{-\frac{a_0}{R''}}\left(\sqrt{t_1 - 3t_1'} - \sqrt{-2t_1'}\right) = \\ = \sqrt{\frac{n+1}{2}\frac{P_\Phi}{Bn}}\sqrt[4]{\frac{-2t_1'}{t_1 - 3t_1'}} + \\ + \frac{\frac{3}{2}R''2t_1' - R''t_1}{\sqrt{\frac{a_0}{2}t_1}\sqrt{\frac{R''}{t_1} + a_0\frac{n+1}{2}\frac{P_\Phi}{Bn}\sqrt{-\frac{a_0}{R''t_1}}}},\end{aligned} \tag{7.10}$$

where $t_1 = t - \tilde{t}'$, $t_1' = t_0 - \tilde{t}'$; (7.10) is used for a qualitative analysis of the motion.

Suppose $t \gg \tilde{t}$. For $t - \tilde{t} \sim \theta^2$, $0 < \alpha < 2$ (7.10) shows that

$$P \sim P_\Phi \theta^{1-\frac{\alpha}{2}}, \quad \theta \sim P_\Phi^{\frac{2}{2+\alpha}}, \quad t - \tilde{t} \sim P_\Phi^{\frac{2\alpha}{2+\alpha}}, \quad P \sim P_\Phi^{\frac{4}{\alpha+2}}. \tag{7.11}$$

These estimates of the orders of magnitude are obtained from the simpler equality $\theta = \sqrt{\frac{n+1}{2}\frac{P}{Bn}}$.

The pressure at A_1 on the shock CA_1 varies from first order for $\alpha = 2$ to second order $\left(\frac{P_\Phi}{Bn}\right)^2$ for $\alpha = 0$. Disturbances at the free surface move along the shock wave with decreasing velocity. The angle θ corresponding to the point A_1 (Figure 15), which separates the zone of surface interference

(θ is measured from the $0x$ axis and relative to an origin at $x = R(\tilde{t})$, $y = 0$), is $\theta \sim P_\Phi^{\frac{2}{2+\alpha}}$; for the arc CA_1 of the perturbed part of the shock wave the order of magnitude of $\theta a_0 (t - \tilde{t}) \sim P_\Phi^{\frac{2-2\alpha}{2+\alpha}}$.

From (4.69) of Chapter I, the velocity of disturbances along the shock front is

$$C(M) \sim \sqrt{\frac{M-1}{2}} \sim (P)^{\frac{1}{2}} \sim P_\Phi^{\frac{2}{2+\alpha}}$$

and for the arc CA_1 the order of magnitude of $C(M)\, a_0 (t - \tilde{t}) \sim P^{\frac{2+2\gamma}{2+\alpha}}$.

It is clear that with the passage of time the rarefaction region on the shock wave increases and becomes first order for finite $t - \tilde{t} > 0$. The flow will be a short wave, where, setting $x - a_0 (t - \tilde{t}) = \delta$, we have

$$P \sim P_\Phi^{\frac{4}{2+\alpha}},\ \delta \sim P_\Phi^{\frac{4}{2+\alpha}},\ \theta \sim P_\Phi^{\frac{2}{2+\alpha}},\ t - \tilde{t} \sim P_\Phi^{\frac{2\alpha}{2+\alpha}}. \tag{7.12}$$

Thus, for $\alpha = 2$ and at time $t \sim \tilde{t}$ we have a short wave of small duration expressing (according to (7.12)) an intense pressure drop with time. However, the ordinary equations of short waves (§ 6) are applicable here, since the time occurs in the form $t_1 \frac{\partial \mu}{\partial t_1}$ and the order of the time does not introduce additional difficulties in the solution. For the problem in the region ACA_1C_1 (Figure 15) we have the boundary conditions on the arc A_1C_1 (the pressure equals that of the linear problem), and a condition for the tangential components /9/.

An approximate solution, valid for a two-dimensional shock wave, is used here, where the flow behind the shock is assumed to be a simple wave /6/. Setting $\theta_{A_1} = \theta_1$, we have along CA_1 (Figure 15)

$$\frac{P}{P_1'} = \frac{1}{4}\left(1 + \frac{\theta}{\theta_1}\right)^2 \tag{7.13}$$

or

$$\theta - \theta_1 = 2\left(\sqrt{\frac{n+1}{2}\frac{P}{Bn}} - \sqrt{\frac{n+1}{2}\frac{P_1'}{Bn}}\right), \tag{7.14}$$

$$P_1' = P_\Phi \sqrt{-\frac{2t_1'}{t_1 - 3t_1'}}, \tag{7.15}$$

where the angle θ_1 is found from (7.10). For finite $t_1 = t - \tilde{t}'$, $\theta_1 = \sqrt{\frac{2R''t'}{a_0}} \sim \frac{P_\Phi}{Bn}$. If $\theta_1 \sim \sqrt{P_1}$, then $\theta_1 \sim \frac{1}{\sqrt{t_1}}$, $P \sim \frac{1}{t_1}$ as $t_1 \to \infty$, and in the axisymmetric problem $P_1' \sim \frac{P_\Phi}{\sqrt{t_1}} \frac{\sqrt{R(t_0)}}{\sqrt{t_1}}$ and $\theta_1 \sim \frac{1}{t_1}$, $P \sim \frac{1}{t_1^2}$. Such attenuation is known for problems with a free boundary /39/.

The theory of the motion of discontinuities along fronts /10/ is now applied to the problem.

For finite t_1

$$\theta = \sqrt{\frac{2R' t_1}{a_0}} + \frac{n+1}{2}\sqrt{-\frac{a_0}{R''}\frac{P_\infty}{Bn}}\frac{1}{\sqrt{t_1}} \tag{7.16}$$

in virtue of (7.6); at A_1

$$\theta_1 = 2\sqrt{-\frac{a_0}{R''}}\frac{n+1}{2}\frac{P_\infty}{Bn}\frac{1}{\sqrt{t_1}}. \tag{7.17}$$

The angle of inclination of the normal to the shock wave in the linear case is

$$\theta' = -\frac{\partial x}{\partial y} = \sqrt{\frac{2R''t_1}{a_0}}. \tag{7.18}$$

Now consider the pressure distribution along the shock wave CA_1, using the theory of the motion of disturbances along a shock wave and assuming a simple-wave flow /10/.

According to § 4 of Chapter I we have, from (4.70),

$$\theta' - \theta_1' = 2\sqrt{\frac{n+1}{2}\frac{P}{Bn}} - 2\sqrt{\frac{n+1}{2}\frac{P_1}{Bn}}. \tag{7.19}$$

where $\theta' = \theta_1'$ and $P = P_1'$ at the point A_1 (Figure 15), and according to (7.10)

$$P_1' = P_\Phi\sqrt{\frac{2t_1}{t_1}}, \quad \theta_1' = \frac{\theta_1}{2}, \quad \theta_1' = \sqrt{\frac{n+1}{2}\frac{P_1'}{Bn}}. \tag{7.20}$$

Equation (7.19) corresponds exactly to (7.15) if $\theta = \theta'$. Thus, the solutions obtained using Riemann waves /6/ and simple waves in the variables $a_0 t_1, \beta$ do not differ in form, except that the angles θ and θ' have different meanings. A relation between the angles θ and θ' is required to completely solve the problem. This relation depends on the form of the shock wave.

From equation (7.18) we have

$$\theta = \theta^1 + \frac{n+1}{2}\sqrt{-\frac{a_0}{R''t_1}\frac{P_\Phi}{Bn}}.$$

According to (7.19), however, such a relation between θ and θ^1 gives $P = 0$ on the surface ($\theta = 0$). Therefore the linear relation (7.18) between θ and θ_1 near the point C (Figure 15) does not apply and, using (7.19), one should find the exact equation of the shock wave /10/. Let the solution for the shock wave conform to /6/

$$\frac{-R(\bar{t}) + x}{a_0 t_1} - 1 = \frac{n+1}{8}\frac{P_1'}{Bn} + \sqrt{\frac{n+1}{8}\frac{P_1'}{Bn}}\frac{y}{a_0 t_1} - \frac{1}{4}\left(\frac{y}{a_0 t_1}\right)^2; \tag{7.21}$$

the relation between θ and $\theta^1 = -\frac{\partial x}{\partial y}$ is then $\theta^1 = -\sqrt{\frac{n+1}{8}\frac{P_1'}{Bn}} + \frac{1}{2}\theta$, and the

pressure at $\theta=0$ is

$$\frac{P}{Bn} = \frac{n+1}{4}\frac{1}{8}\left(\frac{P_\Phi}{Bn}\right)^2 - \frac{a_0}{R''t_1},$$

the pressure at A_1 being expressed by

$$\frac{P_1'}{Bn} = \frac{n+1}{2}\left(\frac{P_\Phi}{Bn}\right)^2 - \frac{a_0}{R''t_1}.$$

If the quadratic term is discarded in (7.21), we obtain agreement with the linear theory of the motion of disturbances along a shock wave /10/.

§8. Pressure at a finite point (x, y) for large times

Consider first an incompressible fluid ($a_0=\infty$) and assume that the pressure P_1 and $R'(t)=V$ are constant.

According to (2.2), which is valid for the axisymmetric problem and an incompressible fluid,

$$P(x, y, t) = -\frac{1}{2\pi}\frac{\partial}{\partial y}\int_0^{2\pi} d\psi \int_0^{Vt} \frac{P_1 r dr}{\sqrt{r^2+x^2-2rx\cos\psi+y^2}}. \tag{8.1}$$

This expression is treated for large times t and finite coordinates x, y.

The value of

$$\frac{\partial P}{\partial t} = P_1 V \frac{1}{2\pi} y \int_0^{2\pi} \frac{Vt d\psi}{(V^2t^2+x^2-2Vtx\cos\psi+y^2)^{3/2}} \tag{8.2}$$

or

$$P_1 V \frac{1}{2\pi}\, y\, \frac{1}{V^2t^2}\int_0^{2\pi} \frac{d\psi}{\left(1+\dfrac{x^2+y^2}{V^2t^2}-2x\cos\psi\dfrac{1}{Vt}\right)^{3/2}} \tag{8.3}$$

is now calculated. If (8.3) is expanded in negative powers of t, we obtain approximately

$$\frac{P}{P_1} = 1 - \frac{y}{Vt} + \frac{y^3}{V^3t^3}\frac{1}{2} - \frac{yx^2}{V^3t^3}\frac{3}{4}. \tag{8.4}$$

Now consider the same problem for an ideal compressible fluid. Here (2.2) yields

$$P(x, y, t) = J_1 + J_2 = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{r_0^*} \frac{P_1 y r dr d\psi}{(x^2+r^2-2xr\cos\psi+y^2)^{3/2}} - \\ - \frac{1}{2\pi}\int_0^{2\pi} \frac{P_1 r_0^* \dfrac{\partial r_0^*}{\partial y}\, d\psi}{\sqrt{r_0^{*2}+x^2-2xr_0^*\cos\psi+y^2}}, \tag{8.5}$$

where

$$r_0^* = \frac{Vt - M^2 x\cos\psi - \sqrt{(Vt - M^2 x\cos\psi)^2 - (1-M^2)(V^2t^2 - x^2M^2 - y^2M^2)}}{1-M^2},$$

$$M = \frac{V}{a_0}. \tag{8.6}$$

For large t and finite x, y

$$\frac{r_0^*}{Vt} = \frac{1}{1+M} + \frac{Mx\cos\psi}{1+M}\frac{1}{Vt} - M\frac{1}{V^2t^2}\frac{x^2+y^2-x^2\cos^2\psi}{2}. \tag{8.7}$$

When J_2 in (8.5) is expanded in powers of $\frac{1}{t}$, we obtain

$$J_2 = P_1 My\frac{1}{Vt} + \frac{P_1}{V^3t^3}(1+M)^2\frac{My}{2}\left(y^2 + \frac{3}{2}x^2\right). \tag{8.8}$$

Proceeding as for the incompressible fluid the following expression is derived from (8.5):

$$\frac{\partial J_1}{\partial t} = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\psi}{r_0^{*2}}\, y\frac{\partial r_0^*}{\partial t} P_1\left(1 - \frac{3}{2}\frac{x^2+y^2}{r_0^{*2}} + 3x\cos\psi\frac{1}{r_0^*} + \frac{15}{8}\frac{4x^2\cos^2\psi}{r_0^{*2}}\right). \tag{8.9}$$

From (8.7) one obtains

$$J_1 = -y\frac{M+1}{Vt}P_1 - \frac{(M+1)^2}{V^3t^3}yM\frac{\frac{1}{2}x^2 + y^2}{6}P_1 -$$

$$- \frac{(M+1)^3\, y - 3x^2 - 3y^2 + \frac{15}{2}x^2}{V^3t^3\quad 6}P_1 + P_1\frac{(M+1)^3}{V^3t^3}My\frac{x^2}{2} + C,$$

where C is determined from the condition $P(x, 0, t) = P_1$. Hence, we finally have

$$\frac{P}{P_1} = 1 - \frac{y}{Vt} + \frac{(M+1)^3}{V^3t^3}y\frac{y^2 - \frac{3}{2}x^2}{2} + \frac{(M+1)^2}{V^3t^3}My\frac{2x^2}{3}. \tag{8.10}$$

Solutions (8.4) and (8.10) show that the influence of compressibility and inhomogeneity only affect powers of $\frac{1}{V^3t^3}$.

§ 9. Pressure in the linear and nonlinear formulations for an inhomogeneous compressible fluid

For an inhomogeneous compressible fluid the boundary-value problem (B) or (C) in the linear case is solved using the principal value of the

integral /27/. The solution in a half-space is found in the form of quadratures of the elementary solution of the inhomogeneous equation together with boundary condition (B) or (C) /27/. Figure 3 shows the disturbed motion when the pressure front velocity over the surface is subsonic, i.e., $R'(t) < a(y)$ ($a(y)$ is the sound velocity distribution with depth), where the equation defining BB' is /27/

$$x = \frac{\sin\theta}{a(0)} \int_0^y \frac{dy}{\sqrt{\frac{1}{a^2(y)} - \frac{\sin^2\theta}{a^2(0)}}}, \qquad t = \int_0^y \frac{dy}{a^2(y)\sqrt{\frac{1}{a^2(y)} - \frac{\sin^2\theta}{a^2(0)}}}, \tag{9.1}$$

which is also the equation of a ray if θ, the angle between the ray and $0y$ at the emergence from the fluid surface (/27/, p. 69), is assumed constant.

Consider the two-dimensional problem. The pressure distribution near the curve (9.1) along the $0y$ axis is given in the form (/27/, p. 84)

$$\frac{P(0, y, t)}{P_A(0)} = \frac{1}{\pi}\sqrt{\frac{a(y)}{a(0)}}\sqrt{\frac{2}{\int_0^y a(y)\,dy}}\,R'(0)\sqrt{t - \tau(0, y)}\int_0^1 f(\xi)d\xi, P_A = P_\Phi, \tag{9.2}$$

where $t = \tau(x, y)$ is equation (9.1). Only the $0y$ axis is treated, although the pressure near the whole of BB' can be fairly easily obtained /27/.

In the supersonic case, the boundary of the disturbed region $A_0B_0C_0$ (Figure 17) consists of the segment B_0C_0 of (9.1) and A_0B_0, a disturbance envelope $t - t_0 = \tau(x - R, y)$ of the form (9.1) with equation

$$x - R(t_0) = \frac{1}{R'(t_0)} \int_0^y \frac{dy}{\sqrt{\frac{1}{a^2(y)} - \frac{1}{R'^2(t_0)}}}, \qquad t - t_0 = \int_0^y \frac{dy}{a^2(y)\sqrt{\frac{1}{a^2(y)} - \frac{1}{R'^2(t_0)}}}. \tag{9.3}$$

The pressure along (9.3) can be found using a ray representation /27/.

Suppose the propagation velocity of the pressure over the surface is $R'(t) = V$, a constant, and we assume $a(y)$ is an increasing function of y, where $a(0) < V$ at the surface.

Now consider the behavior of the disturbed fronts. Suppose $a(y_1) = V$ for some y_1. At times for which the front has not reached $y = y_1$, the front has its usual form A_0B_0 (Figure 17). The point of contact of an elementary wave B_0C_0 with the front envelope A_0B_0 is given by

$$\frac{\sin\theta_1}{a_0} = \frac{1}{V}, \quad a_0 = a(0). \tag{9.4}$$

The angle of turn of the front B_0C_0 is determined /24/ by

$$\frac{\sin \vartheta_2}{a(0)} = \frac{1}{a(y)}, \tag{9.5}$$

which corresponds to the point beyond which the ray (9.1) lies on the descending branch and the front (9.1) turns (see Figure 17, vicinity of the point K) /32/.

When $y < y_1$, then $a(y) < V$ and $\vartheta_1 < \vartheta_2$, i.e., the point of closest approach of the front (9.1) lies below the point of contact, and the shock front is given by $A_0B_0C_0$ (Figure 17).

For $a(y) = V$ the point of contact B_0 lies on $y = y_1$; then, $\vartheta_1 = \vartheta_2$ and the point of closest approach of the front B_0C_0 lies on $a(y) = V$.

When $a(y) < V$ and $y > y_1$, the front turns. At the point $y = y_1$ the curve AB has a break point E (Figure 17), corresponding (according to (9.3)) to the turn of the front. The curve BC has a point of closest approach corresponding to $\frac{\sin \vartheta_2}{a(0)} = \frac{1}{a(y)}$, where, since $a(y) > V$ and $\vartheta_2 < \vartheta_1$, this point is situated higher than the point of contact B (Figure 17). The point of contact on the ray $\frac{\sin \vartheta_1}{a(0)} = \frac{1}{V}$ is already situated on the descending branch of the ray. The velocity of the point E is $\frac{dx}{dt} = V$. The velocity of the point K is $\frac{dx}{dt} = \frac{a(y_1)}{\cos \alpha}$, where α is the angle between the normal to the front and $0x$, given by

$$\operatorname{ctg} \alpha = -\frac{\sin \vartheta}{a(0)} \frac{1}{\sqrt{\frac{1}{a^2(y_1)} - \frac{\sin^2 \vartheta}{a^2(0)}}}.$$

Hence $\frac{dx}{dt} = \frac{a(0)}{\sin \vartheta} > V$, since $\sin \vartheta < \frac{a(0)}{V}$ (the fronts are illustrated in Figure 17).

The curve $y = y_1$ corresponds to the case of total internal reflection for A_0B_0. The solution on the major portion of the front up to its turn is given in /27/.

Consider the case $R'(0) = \infty$. Here, the envelope (9.3) reaches the $0y$ axis. Suppose $R'(t) > a(0)$ on the surface $y = 0$, but at some depth $a(y) > R'(t)$. Near the $0y$ axis $R'(t_0) > a(y)$ always; in particular on the axis itself, when $t_0 = 0$, $R'(0) = \infty$. However, on some segments of the front, beginning at some instant, the condition $R'(t_0) \leqslant a(y)$ will be satisfied, i.e., the horizontal velocity of the front becomes subsonic. It is of interest to determine the curve $x^* = x(y^*)$ along which $R'(t_0) = a(y^*)$. If t_0 and t are eliminated from here and (9.3), the equation of the curve $x^* = x(y^*)$ results, after which the front begins to turn.

Consider now a point disturbance (this is treated as in /24/), and suppose the sound velocity varies linearly with depth, where $a(y) = a_0(1 + \varepsilon y)$, $a_0 = a(0)$. The motion $R(t)$ of the front over the surface is taken as $R(t) = D\sqrt{t^2 + 2tt_0'}$, $A = D\sqrt{2t_0'}$ (§1, Chapter II), or $R(t) = A\sqrt{t}$ for small times.

The calculations are almost identical for any $R(t)$. The equation of the line along which the front becomes parallel to the $0y$ axis can be found from the condition $a(y^*) = R'(t_0)$ and (9.3), where

$$1 + \varepsilon y^* = \frac{R'(t_0)}{a_0},$$

$$x^* - R(t_0) = \frac{1}{\varepsilon}\sqrt{\frac{R'^2(t_0)}{a_0^2} - 1}.$$

Using the equality $R(t_0) = A\sqrt{t_0}$ and eliminating t_0, we find

$$x^* = \frac{A^2}{2a_0(1+\varepsilon y^*)} + \frac{1}{\varepsilon}\sqrt{(1+\varepsilon y^*)^2 - 1}.$$

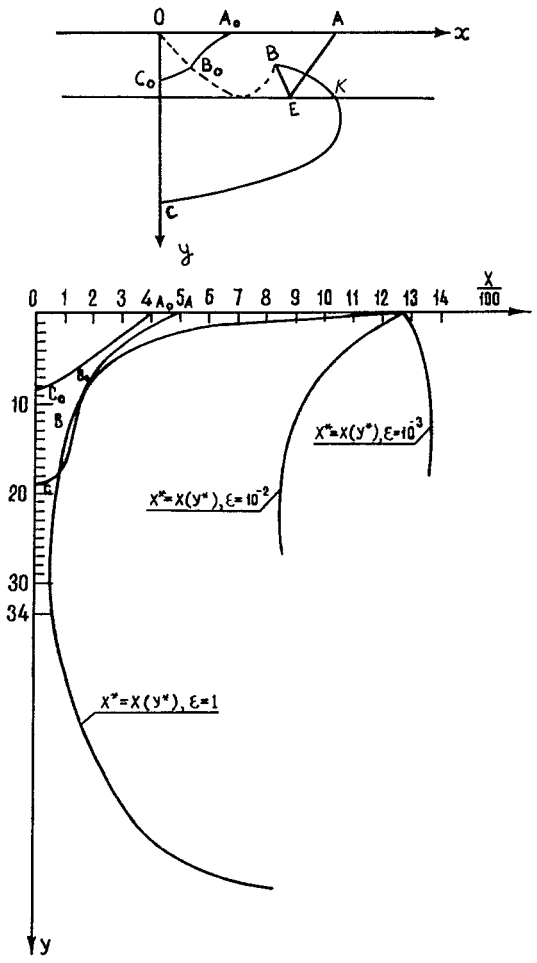


FIGURE 17, 17'.

The curve is shown in Figure 17; it has an extremum for $\frac{dx^*}{dy^*}=0$. In the calculations we set $A = 2{,}000\ \text{msec}^{-\frac{1}{2}}$, $a_0 = 1{,}540\ \text{msec}^{-1}$, and $\varepsilon = 10^{-3}$, 10^{-2}, 1. When $\varepsilon = 10^{-3}$ (9.3) intersects $x^* = x(y^*)$ once; for $\varepsilon = 1$ and 10^{-2} there are two points of intersection, $y_{1,2}$. The front has the form $A_0B_0C_0$ (Figure 17') until it intersects $x^* = x(y^*)$; thereafter, the segments of the front for $y < y_{1\,\text{inter}}$ and $y > y_{2\,\text{inter}}$ are again given by (9.3). The first point of intersection in time corresponds to the value $x^* = x^*_{\min}$, $\frac{dx^*}{dy^*}=0$. When $y_1 < y < y_2$, the equation of the ray is given by the descending branch $(x > x^*)$

$$x - R(t_0) = \frac{1}{\varepsilon}\sqrt{\frac{R'^2(t_0)}{a_0^2} - 1} + \frac{1+\varepsilon y}{\varepsilon}\sqrt{\frac{R'^2}{\frac{a_0^2}{(1+\varepsilon y)^2}} - 1},$$

and t, which is given by an expression differing from (9.3), is

$$t = t_0 + \int_0^{y^*} \frac{dy}{a^2(y)\sqrt{1/a^2(y) - 1/R'^2(t_0)}} - \int_{y^*}^{y} \frac{dy}{a^2(y)\sqrt{1/a^2(y) - 1/R'^2(t_0)}}.$$

In contrast to the case $R'(t)=\text{const}$, for $y = y^*$ on $x^* = x(y^*)$ the radius of curvature of $A_0B_0C_0$ does not vanish along the front $x(y)$; $\frac{dx}{dy} = \mp R'\sqrt{\frac{1}{a^2(y)} - \frac{1}{R'^2}}$, $\left(\frac{dx}{dy}\right)_{y=y^*}=0$, $\frac{d^2x}{dy^2} \sim 0(1)$; and no singularity exists. Consider again the case $R'(t)=\text{const}$. For the wave front segment BE (Figure 17) which passes the caustic $y = H$, we have, according to /23/,

$$\frac{P}{P_1^0} = \sqrt[4]{\frac{1/a^2(0) - 1/V^2}{1/a^2(y) - 1/V^2}}\ln(t - t_0'),$$

where $t = t_0'$ corresponds to the front BE. In order to eliminate the singularity at $y = H$, we apply the consideration /10/ (see also § 7) that focusing is eliminated when the linear characteristics are replaced by the exact ones. The equation of the improved characteristics is given by the first equation of (9.3), and the second condition of (9.3) is replaced by

$$t - t_0 = \int_0^{y} \frac{dy}{a^2(y)\sqrt{\frac{1}{a^2(y)} - \frac{1}{V^2}}} - \alpha_0 \int_0^{y} \frac{P\,dy}{P(y)\,a^2(y)\sqrt{1/a^2(y) - 1/V^2}} + \text{const}.$$

Near $y=H$, $a(y) - V$ is of the order of $H - y$, $P = \frac{P_1^0 C}{\sqrt[4]{H-y}}$ and C is constant. The inclination of the front to $0y$ and its radius of curvature are derived from $\frac{dx}{dy} = \sqrt{H-y}$, $\frac{d^2x}{dy^2} = \frac{1}{\sqrt{H-y}}$ in the linear case and $\frac{dx}{dy} = \frac{P_1^0}{(H-y)^{\frac{3}{4}}}$, $\frac{d^2x}{dy^2} = \frac{P_1^0}{(H-y)^{\frac{7}{4}}}$ in the nonlinear case. When these expressions are equated

near the caustic, $(H-y)^{\frac{5}{4}} = P_1^0$, $P = P_1^{0\,\frac{4}{5}}$, $V_x = P_1^{0\,\frac{4}{5}}$, $V_y = V_x \frac{dx}{dy}$. If $x - Vt = -V\int_0^y \sqrt{\frac{1}{a^2(y)} - \frac{1}{V^2}}\,dy$ near the front BE (Figure 17) in the vicintiy of the point E, and we introduce the variable x' defined by $x - Vt - x' = -V\int_0^H \sqrt{\frac{1}{a^2(y)} - \frac{1}{V^2}}\,dy$, we easily see that $x' = \pm V\int_y^H \sqrt{\frac{1}{a^2(y)} - \frac{1}{V^2}}\,dy$, $x' \sim P_1^{0\,\frac{6}{5}}$, $y' = y - H$ in the neighborhood of the wave front. The solution is now found near the front in the nonlinear problem in the vicinity of the caustic. Making use of the approximations, equations (F) can be written in Cartesian coordinates x', y, t. If small quantities of order $-\frac{2}{5}$ and $\frac{2}{5}$ are retained in the first and third equations and small quantities of zero order in $\frac{P_1}{Bn}$ are retained in the second equation, then after eliminating P we obtain

$$-\frac{\partial V_x}{\partial x}(V - V_x)^2 + a^2\frac{\partial V_x}{\partial x'} + a_1^2\frac{\partial V_y}{\partial y'} = 0,$$

$$\frac{\partial V_x}{\partial y'} = \frac{\partial V_y}{\partial x'},$$

where $a^2 = a^2(y) + 2(\varkappa^\circ - 1)\,a_1 V_x$ from (9.6) (see below) and $V = a_1$, the velocity of sound in the undisturbed fluid at $y = H$, i.e., on the caustic. The characteristics of this system of equations have the form

$$-\left(\frac{dx'}{dy'}\right)^2 = 2a_1'\frac{y'}{a_1} + 2(\varkappa^\circ + 1)\frac{V_x}{a_1},\quad dV_y = \frac{dx'}{dy'}\,dV_x.$$

The motion is steady state in terms of the variables x', y'. In the linear case we have

$$\frac{dx'}{dy'} = -\sqrt{-2a_1'\frac{y'}{a_1}},\quad V_y = -V_x\frac{dx'}{dy'}$$

and

$$2\frac{dV_x}{dy'} + \frac{1}{\frac{dx'}{dy'}}\frac{d^2x'}{dy'^2}\,V_x = 0$$

along the front, and by integrating we obtain the solution on the front /27/, $V_x = \frac{P_1}{\rho_0 a_1}\sqrt[4]{\frac{1/a_0^2 - 1/V^2}{1/a^2(y) - 1/V^2}}$, where

$$\sqrt{\frac{1}{a^2(y)} - \frac{1}{V^2}} = \sqrt{\frac{2a_1'}{a_1^3}}\sqrt{-y'}$$

near point E. If $V_y = -V_x\frac{dx'}{dy'}$ in the equations of the characteristics for the nonlinear case, we have

$$2\frac{dV_x}{dx'} - \frac{1}{\frac{dx}{dy'}}\frac{d^2x'}{dy'^2}\,V_x = 0$$

which, after integration, yields

$$V_x = \frac{C}{\sqrt[4]{-2\frac{a_1'}{a_1}y' - 2(\alpha^\circ + 1)\frac{V_x}{a_1}}},$$

where C is derived from the linear solution in the form

$$C = \frac{P_1}{\rho_0 a_1}\sqrt[4]{\frac{1}{a_0^2} - \frac{1}{V^2}}\sqrt{a_1}.$$

The obtained solution is nowhere equal to ∞, since, if

$$Z = \sqrt[4]{-2\frac{a_1'}{a_1}y' - 2(\alpha^\circ + 1)\frac{P_1}{\rho_0 a_1}\frac{\sqrt[4]{1/a_0^2 - 1/V^2}\sqrt{a_1}}{\sqrt[4]{-2\frac{a_1'}{a_1}y' - \cdots}}},$$

we obtain

$$Z = \sqrt[4]{-2\frac{a_1'}{a_1}y' - \frac{2(\alpha^\circ + 1)\frac{P_1}{\rho_0 a_1}}{Z}\sqrt[4]{1/a_0^2 - 1/V^2}\sqrt{a_1}},$$

which does not have a root $Z = 0$. The same result is also obtained by Lighthill's technique.

To obtain a more exact solution, consider first the linear case. The system of equations

$$\frac{\partial v_y}{\partial x'} - \frac{\partial v_x}{\partial y'} = 0,$$

$$\frac{\partial v_x}{\partial x'}(\alpha^\circ v_x + a'y') + \frac{V}{2}\frac{\partial v_y}{\partial y'} = 0$$

reduces to the system

$$v_x = \frac{\partial \varphi_0}{\partial x}, \quad v_y = \frac{\partial \varphi_0}{\partial y},$$

$$y'\frac{\partial^2 \varphi_0}{\partial x'^2} + \frac{V}{2a'}\frac{\partial^2 \varphi_0}{\partial y'^2} = 0$$

after discarding the first term in parentheses. The solution of this equation, which corresponds to the problem of the reflection of a weak discontinuity, or of the front BE (Figure 17) from the sonic line $y' = 0$, is given in /13/. In our case there is a discontinuity in v_x and the solution of /13/ should be taken for $k = 5/12$. Hence, near BE

$$\varphi_0 = Ax'^{\frac{5}{6}}\left(1 + \frac{4y'^3}{9x'^2\frac{V}{2a'}}\right)F_0\left(\frac{7}{12}, \frac{13}{12}, 2, 1 + \frac{4y'^3}{9x'^2\frac{V}{2a'}}\right).$$

where $\Phi = 0$ for $x' < 0$; for $x' > 0$ we have, from the condition on the front BE, where $1 + \frac{4y'^3}{9x'^2 \frac{V}{2v'}} = 0$, and by comparing $\frac{\partial \varphi_0}{\partial x}$ with v_x on BE,

$$A \frac{16}{9} \frac{a'}{V} \left(\frac{8a'}{9V}\right)^{-\frac{13}{12}} = \frac{P_1^0}{\rho_0 V} \frac{\sqrt[4]{\frac{1}{a_0^2} - \frac{1}{V^2}} \sqrt{V}}{\sqrt[4]{\frac{2a'}{V}}}.$$

The solution near the reflected front is obtained by analytic continuation /13/ and has the form

$$\varphi_0 = A(-x')^{\frac{5}{6}} \left\{ \frac{\sin \pi \left(4k - \frac{1}{6}\right)}{\sin \pi \left(2k + \frac{1}{6}\right)} \left| 1 + \frac{4y'^3}{9x'^2 \frac{V}{2a'}} \right| \times \right.$$

$$\times F\left(\frac{7}{12}, \frac{13}{12}, 2, 1 + \frac{4y'^3}{9x'^2 \frac{V}{2a'}}\right) + 2^{4k + \frac{4}{3}} \cos \pi \left(2k + \frac{1}{6}\right) \frac{\Gamma\left(3k + \frac{1}{6}\right) \Gamma\left(2k + \frac{7}{6}\right)}{\Gamma(2k+1)\Gamma\left(2k + \frac{1}{3}\right)} \times$$

$$\left. \times F\left(-k, -k + \frac{1}{2}, 2k + \frac{5}{6}, 1 + \frac{4y'^3}{9x'^2 \frac{V}{2a'}}\right) \right\}.$$

For $k = {}^5/_{12}$ both terms have a singularity. Setting $k = {}^5/_{12} + \varepsilon$, it is found that terms in $\frac{1}{\varepsilon}$ cancel; near the reflected front $x' = -\frac{2}{3}\left(\frac{2a'}{V}\right)^{\frac{1}{2}}\left(-y'\right)^{\frac{3}{2}}$ we have approximately

$$\varphi_0 = \frac{1}{\pi}(-x')^{\frac{5}{6}} A \left\{ -\left(1 + \frac{4y'^3}{9x'^2 \frac{V}{2a'}}\right) \ln \left| 1 + \frac{4y'^3}{9x'^2 \frac{V}{2a'}} \right| + \right.$$

$$+ \frac{1}{2}\left(1 + \frac{4y'^3}{9x'^2 \frac{V}{2a'}}\right) - \frac{144}{5} + \frac{2329}{5184}\left(1 + \frac{4y'^3}{9x'^2 \frac{V}{2a'}}\right)^2 +$$

$$\left. + \left(1 + \frac{4y'^3}{9x'^2 \frac{V}{2a'}}\right)\left(2\gamma - 2\ln 2 + \psi\left(\frac{5}{6}\right) + \psi\left(\frac{1}{6}\right)\right) \right\};$$

$$\psi(x) = \Gamma'(x)/\Gamma(x); \quad \gamma = \Gamma'(1)/\Gamma(1).$$

As expected, a discontinuity in P or v_x on the incident front BE gives rise to a logarithmic singularity on the reflected front.

In the nonlinear problem the following notation is introduced:

$$v_{x_1} = \frac{\partial \psi}{\partial x'}, \quad v_{y_1} = \frac{\partial \psi}{\partial y'}, \quad \psi = \frac{\alpha^\circ}{a'} \varphi + x'y';$$

hence, an equation for ψ is obtained in the form

$$\frac{\partial^2 \psi}{\partial x'^2} v_{x_1} + \frac{V}{2a'} \frac{\partial^2 \psi}{\partial y'^2} = 0,$$

which, after the Legendre transformation

$$\Phi + \varphi = x'v_{x_1} + y'v_{y_1},\ x' = \frac{\partial \Phi}{\partial v_{x_1}},\quad y' = \frac{\partial \Phi}{\partial v_{y_1}},$$

yields an equation for the linear problem given by

$$v_{x_1}\frac{\partial^2 \Phi}{\partial v_{x_1}^2} + \frac{V}{2a'}\frac{\partial^2 \Phi}{\partial v_{y_1}^2} = 0,$$

where $v_{x_1} = y'$, $v_{x_1} = x'$ for finite x', y'.

In order that φ, which is expressed in terms of the solution of this equation, transforms to φ_0 for finite x', y', it is sufficient to set

$$\Phi = -\frac{\alpha^\circ}{a'}A(v_{y_1})^{\frac{5}{6}}\left(1 + \frac{4v_{x_1}^3}{9v_{y_1}^2\frac{V}{2a'}}\right)F\left(\frac{7}{12}, \frac{13}{12}, 2,\ 1 + \frac{4v_{x_1}^3}{9v_{y_1}^2\frac{V}{2a'}}\right) + v_{x_1}y_{y_1}.$$

The potential $\varphi \sim \left(\frac{P_1^0}{Bn}\right)^2$ and the velocity component tangential to the shock $\sim \left(\frac{P_1^0}{Bn}\right)^{\frac{6}{5}}$ are equal to zero on the shock wave in the given approximation. The solution of the problem has the form

$$x' = \frac{\partial \Phi}{\partial v_{x_1}},\quad y' = \frac{\partial \Phi}{\partial v_{y_1}},$$

where v_x and v_y are found by solving the obtained system. Near the reflected wave, the solution is found (as previously for the linear case) by analytic continuation of the hypergeometric function, where in place of $\varphi_0(x', y')$ we obtain φ expressed in terms of

$$\Phi = -\frac{\alpha^\circ}{a'}\varphi_0(v_{y_1}, v_{x_1}).$$

The shock wave is found from the solution of

$$\frac{dx'}{dy'} = -\sqrt{-\frac{a'}{V}y' - \alpha^\circ v_x},$$

where $v_x(x', y')$ is determined from the reduced system.

These ideas can be applied to steady-state flow with small disturbances in v_x, v_y. The relevant equations are

$$\frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x},$$

$$\{(n+1)v_x + M_\infty^2 - 1\}\frac{\partial v_x}{\partial x} + \frac{\partial v_x}{\partial y} = 0,$$

where M_∞ is the Mach number of the incident flow. A solution of this system is sought such that it is linear in the hodograph plane and transforms to the known solutions for finite $M_\infty - 1$.

Let us now determine the pressure on the shock wave BB' (Figure 3) or B_0C_0 (Figure 17) in the second approximation for the two-dimensional problem. For this one requires an equation of state for an inhomogeneous fluid, since the equation of state (A) is valid for a homogeneous fluid. Let $V=\frac{1}{\rho}$ be the specific volume. The velocity of sound and the parameters can then be found as a simple traveling wave /13/, or, along the characteristic

$$a+u_1=a_0+\alpha^\circ u_1 \tag{9.6}$$

to first order, where

$$\alpha^\circ=\frac{a_0^4}{2V_0^3}\left(\frac{\partial^2 v}{\partial P^2}\right)_{s_1},\quad V_0=\frac{1}{\rho_0},\quad a_0^2=\left(\frac{\partial P}{\partial \rho}\right)_s, a_0=a(y).$$

Since in addition $v_1=a_0\left(\frac{P}{\rho_0 a_0^2}\right)$, we have

$$a+v_1=a_0\left(1+\alpha^\circ\frac{P}{\rho_0 a_0^2}\right).$$

For a homogeneous fluid (A) gives

$$\alpha^\circ=\frac{n+1}{2}.$$

The solution near B_0C_0 is now written in the form /27/

$$\frac{P(x,y,t)}{P_\Phi(0)}=\frac{1}{\pi}\sqrt{\frac{\mu(0)\,2x}{\frac{\mu(y)}{J_1(y,0)}\frac{\lambda}{J_3(y,0)}}R'^2(0)}\int_{-1}^{1}\frac{f(\xi)}{(1-R'\lambda\xi)^2}\,d\xi\,\sqrt{t-\tau(x,y)}, \tag{9.7}$$

where $\lambda=\frac{\sin\vartheta}{a(0)}$, the equation $t=\tau(x,y)$ is given by (9.1), and

$$J_1(y,0)=\int_y^0\frac{dy}{a^2(y)\,\mu^3(y)},$$

$$J_3(y,0)=\int_y^0\frac{dy}{\mu(y)},\quad \mu(y)=\sqrt{\frac{1}{a^2(y)}-\lambda^2}.$$

According to formula (B) the function f gives the pressure at the surface behind the front (/27/, p. 83).

If s denotes the length of the arc along a characteristic ray, $\frac{\partial t}{\partial s}=\frac{1}{a+v_1}$

and

$$\frac{\partial t}{\partial s}=\frac{1}{a(y)}-\frac{1}{a(y)}\,\alpha^\circ\frac{P}{\rho_0 a^2(y)} \tag{9.8}$$

along a characteristic. Integrating (9.8) along a characteristic, we find

$$t=\int \frac{ds}{a(y)} -\alpha^{\circ} \int \frac{ds}{a(y)} \frac{P}{\rho_0 a^2(y)} + y_1. \tag{9.9}$$

Here, P is found from (9.7) and we replace $t-\tau(x, y)$ by y_1, where $y_1=\text{const}$ is the equation of the characteristics (9.9).

When (9.9) is differentiated along the shock front and the formula for the front velocity

$$\frac{\partial t}{\partial s}=\frac{1}{a(y)}-\frac{\alpha^{\circ}}{2}\frac{1}{a(y)}\frac{P}{\rho_0 a^2(y)}$$

is used, an equation for y_1 along the front is derived in the form

$$y_1\int_0^s \frac{ds}{a^2(y)}\frac{P\sqrt{y_1}}{\rho_0 a^2(y)}=\frac{2}{\alpha^{\circ}}\int_0^y \sqrt{y_1}\,dy_1. \tag{9.10}$$

Equation (9.10) can also be obtained from the relations for short waves /18/.

Substituting the expression for P from (9.7) in (9.10) and using the fact that $ds=\dfrac{dy}{a(y)\mu(y)}$ along a ray, the distribution of y_1 along the shock wave is found to be

$$\sqrt{y_1}\,\frac{4}{3}\frac{1}{\alpha^{\circ}}=\frac{R'(0)}{\pi}\int_0^y \frac{C_1\sqrt{\mu(0)}\,\dfrac{P_{\Phi}(0)}{\rho_0 a^2(y)}\sqrt{\dfrac{2x}{\lambda}}\,dy}{a^2(y)\,\mu^3(y)\sqrt{J_1(y,0)\,J_2(y,0)}}, \tag{9.11}$$

where

$$C_1=\frac{1}{2}\int_{-1}^{1}\frac{f(\xi)}{(1-R'\lambda\xi)^2}\,d\xi.$$

If (9.11) is substituted in (9.7), the pressure along the shock wave B_0C_0 is found to be a second-order small quantity with respect to $\dfrac{P_{\Phi}}{Bn}$.

The method of replacing the linear characteristics by improved ones is hence also effective for the case of an inhomogeneous fluid.

The equations of state for an inhomogeneous fluid given in /38/ enable the coefficient α° to be calculated for a simple wave.

To substantiate the obtained solution one must check that the equations of motion are satisfied (as in § 6, Chapter I).

The variables s, ϑ are introduced in place of x, y (cf. (9.1)), where $ds=a(y)\,dt$. Since the derivatives with respect to ϑ are (according to (9.7)) of a higher order of smallness (the condition of one-dimensionality in the region BB'; see § 4, Chapter I), one should check the equation of motion

and the equation of continuity in the direction s of the ray:

$$\frac{\partial v_s}{\partial t} + v_s \frac{\partial v_s}{\partial s} = -\frac{1}{\rho}\frac{\partial P}{\partial s},$$

$$\frac{\partial P}{\partial t} + v_s \frac{\partial P}{\partial s} + \rho a^2 \frac{\partial v_r}{\partial s} + \rho a^2 \frac{v_s}{H_2}\frac{\partial H_2}{\partial s} = 0, \qquad (9.12)$$

where

$$H_2 = \int_0^y \frac{dy}{a^2(y)\left(\frac{1}{a^2(y)} - \frac{\sin^2\vartheta}{a^2(0)}\right)^{1/2}} \frac{\cos\vartheta}{a(0)} \sqrt{1 - a^2(y)\frac{\sin^2\vartheta}{a^2(0)}}.$$

The first equation, as for a homogeneous fluid, is satisfied to order 0 (1) and yields $v_s = -\frac{P}{\rho(y)\,a(y)}$. An application of Lighthill's technique in § 3 to the present problem gives a correct value for the solution on the limiting line, but on the whole the solution is incorrect. To solve the problem we apply the method of slowly varying amplitude /45/, which is very similar to the previous methods. The $0y$ axis will be considered, although the solution will also be correct for BB'. It follows from the solution (9.2) of the linear problem that the dependence of P on the variable $\tau \sim t - \int_0^y \frac{dy}{a(y)}$ is more important near the front BB', where $\tau \sim 0$, than the dependence on the variable y. Therefore, if we transform equations (9.12) to the variables τ, y, they become (on the $0y$ axis for $\vartheta = 0$)

$$\frac{\partial v_y}{\partial \tau} + v_y\left(\frac{\partial v_y}{\partial y} - \frac{\partial v_y}{\partial \tau}\frac{1}{a(y)}\right) = -\frac{1}{\rho}\left(\frac{\partial P}{\partial y} - \frac{\partial P}{\partial \tau}\frac{1}{a(y)}\right),$$

$$\frac{\partial P}{\partial \tau} = \rho(y)\,a(y)\frac{\partial v_y}{\partial \tau} + 2a^\circ v_y \rho(y)\frac{\partial v_y}{\partial \tau} - \rho(y)\,a^2(y)\frac{\partial v_y}{\partial y} - \rho(y)\,a^3(y)\,v_y \Big/ \int_0^y a(y)\,dy, \qquad (9.13)$$

where, in the derivatives with respect to y for constant τ, $P = \rho(y)\,a(y)\,v_y$ and it was taken into account that $\rho a^2 = \rho(y)\,a^2(y) + (2a^\circ - 1)\,\rho(y)\,a(y)\,v_y$, $\rho = \rho(y) + \rho(y)\frac{v_y}{a(y)}$. An equation for $v_y(\tau, y)$ is obtained easily in the form

$$-2a^\circ v_y \frac{\partial v_y}{\partial \tau} + 2a^2(y)\frac{\partial v_y}{\partial y} + v_y \frac{a(y)}{\rho(y)}\frac{\partial \rho(y)\,a(y)}{\partial y} + \frac{a^2(y)\,v_y}{\int_0^y a(y)\,dy} = 0.$$

The characteristics of this equation, namely

$$\frac{d\tau}{-2a^\circ v_y} = \frac{dy}{2a^2(y)} = -\frac{dv_y/v_y}{a^3(y)/\int_0^y a(y)\,dy + a(y)/\rho(y)\cdot\partial\rho(y)\,a(y)/\partial y}$$

can be integrated to yield

$$v_y \sqrt{\rho(y)\, a(y) \int_0^y a(y)\, dy} = C,$$

$$\tau = -Ca^2 \int_0^y \frac{dy}{a^2(y) \sqrt{\rho(y)\, a(y) \int_0^y a(y)\, dy}} + C_1.$$

The general solution is found in terms of an arbitrary function defined by

$$\tau + x^2 v_y \sqrt{\rho(y)\, a(y) \int_0^y a(y)\, dy} \int_0^y \frac{dy}{a^2(y) \sqrt{\rho(y)\, a(y) \int_0^y a(y)\, dy}} =$$

$$= F\left(v_y \sqrt{\rho(y)\, a(y) \int_0^y a(y)\, dy} \right). \qquad (9.14)$$

The function F is determined from the condition that far from the front BB' the first term on the left-hand side of the above equation considerably exceeds the second, and the linear solution (9.2) holds. When one takes into account the variability of the initial density $\rho(y)$ and the relation $v_y = \dfrac{P}{\rho(y)\, a(y)}$ we have

$$v_y = \frac{\sqrt{2}}{\pi} \frac{P_A(0)}{\sqrt{\rho(0)\, a(0)}} V \int_{-1}^{1} f(\xi)\, d\xi \quad \frac{\sqrt{\tau}}{\sqrt{\rho(y)\, a(y) \int_0^y a(y)\, dy}},$$

$$V = R'(0).$$

These expressions result in the relation

$$F(x) = C_3^2 x^2, \; C_3 = \frac{\sqrt{\rho(0)\, a(0)}}{P_A(0)} \frac{\pi}{\sqrt{2}\, V \int_{-1}^{1} f(\xi)\, d\xi}$$

and the solution in the form

$$\tau + x^2 v_y \sqrt{\rho(y)\, a(y) \int_0^y a(y)\, dy} \int_0^y \frac{dy}{a^2(y) \sqrt{\rho(y)\, a(y) \int_0^y a(y)\, dy}} =$$

$$= C_3^2 v_y^2 \rho(y)\, a(y) \int_0^y a(y)\, dy.$$

The solution over the whole of BB' is obtained by replacing y by s and $\int_0^y a(y)\, dy$ by H_2. The obtained solution agrees with (9.7) and (9.9), found by Whitham's method. The solution near the shock wave for an inhomogeneous

fluid for large times, which can be obtained from the foregoing procedure, is described in Appendix III.

Consider now the equation in the linear case /27/

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = \frac{1}{a^2(y)} \frac{\partial^2 P}{\partial t^2} \tag{9.15}$$

for the two-dimensional problem. The results for the axisymmetric problem are derived from the solution for the two-dimensional one by substituting $P' = P\sqrt{x}$. To determine the pressure in a region of disturbed motion far from the wave front $A_0B_0C_0$ (Figure 17), the solution was represented in powers of a parameter k, which characterizes the inhomogeneity of the fluid /27/, where

$$\frac{1}{a^2(y)} = \frac{1}{a_0^2} + ka_1 y + \cdots.$$

For example, the value of P_1 in the first approximation for the axisymmetric problem is expressed in terms of the solution of the homogeneous problem ($k = 0$):

$$\frac{P_1}{\frac{1}{4} a_1} = y^2 \int_{\infty}^{y} \frac{\partial^2 P_0}{\partial t^2}\, dy - y \int_{\infty}^{y} \int_{\infty}^{y} \frac{\partial^2 P_0}{\partial t^2}\, dy dy. \tag{9.16}$$

This solution, however, is incorrect near the front $A_0B_0C_0$. In order to construct a solution which is correct everywhere, we follow the procedure in §4 of Chapter I and set

$$\frac{\partial \varphi}{\partial t} = P, \quad \varphi = \sum_1^{\infty} k^{n-1} \varphi_n(u, x, y),$$
$$t = u + \sum_1^{\infty} k^n t_n(x, y), \quad t_a = \sum_1^{\infty} k^n t_n(x, y), \tag{9.17}$$

where φ is a function which vanishes on $A_0B_0C_0$. For simplicity, consider the linear distribution

$$\frac{1}{a^2(y)} = \frac{1}{a_0^2} + a_1 k y.$$

Since

$$\left.\frac{\partial \varphi}{\partial x}\right|_t = \left.\frac{\partial \varphi}{\partial x}\right|_u - \frac{\partial \varphi}{\partial u} \frac{\partial t_a}{\partial x} \tag{9.18}$$

with similar expressions for the other derivatives, we have from (9.15)

$$\sum_1^{\infty} k^{n-1} \left(\frac{\partial^2 \varphi_n}{\partial x^2} + \frac{\partial^2 \varphi_n}{\partial y^2}\right) - 2 \sum_1^{\infty} k^n \frac{\partial t_n}{\partial x} \sum_1^{\infty} k^{n-1} \frac{\partial^2 \varphi_n}{\partial u \partial x} -$$
$$- 2 \sum_1^{\infty} k^n \frac{\partial t_n}{\partial y} \sum_1^{\infty} k^{n-1} \frac{\partial^2 \varphi_n}{\partial u \partial y} + \left(\sum_1^{\infty} k^n \frac{\partial t_n}{\partial x}\right)^2 \sum_1^{\infty} k^{n-1} \frac{\partial^2 \varphi_n}{\partial u^2} +$$
$$+ \left(\sum_1^{\infty} k^n \frac{\partial t_n}{\partial y}\right)^2 \sum_1^{\infty} k^{n-1} \frac{\partial^2 \varphi_n}{\partial u^2} - \sum_1^{\infty} k^{n-1} \frac{\partial \varphi_n}{\partial u} \sum_1^{\infty} k^n \left(\frac{\partial^2 t_n}{\partial x^2} + \frac{\partial^2 t_n}{\partial y^2}\right) = \left(\frac{1}{a_0^2} + a_1 k y\right) \sum_1^{\infty} k^{n-1} \frac{\partial^2 \varphi_n}{\partial u^2}. \tag{9.19}$$

Setting $t_n = 0$ and $\varphi_n = 0$ for $n \leqslant 0$ we write (9.19) in the form

$$\frac{\partial^2 \varphi_n}{\partial u^2} - a_0^2 \frac{\partial^2 \varphi_n}{\partial x^2} - a_0^2 \frac{\partial^2 \varphi_n}{\partial y^2} = -2a_0^2 \sum_1^{n-1} \left(\frac{\partial t_i}{\partial x} \frac{\partial^2 \varphi_{n-i}}{\partial u \partial x} + \frac{\partial t_i}{\partial y} \frac{\partial^2 \varphi_{n-i}}{\partial u \partial y} \right) +$$

$$+ a_0^2 \sum_1^{n-2} \frac{\partial^2 \varphi_{n-i}}{\partial u^2} \sum_1^{i-1} \left(\frac{\partial t_{i1}}{\partial x} \frac{\partial t_{i-i_1}}{\partial x} + \frac{\partial t_{i1}}{\partial y} \frac{\partial t_{i-i_1}}{\partial y} \right) - a_0^2 a_1 y \frac{\partial^2 \varphi_{n-1}}{\partial u^2} - a_0^2 \sum_1^{n-1} \frac{\partial \varphi_{n-i}}{\partial u} \left(\frac{\partial^2 t_i}{\partial x^2} + \frac{\partial^2 t_i}{\partial y^2} \right). \tag{9.20}$$

The right-hand side of (9.20) will be denoted by f_n.

Equation (9.20) can be solved with the aid of formula (4.51) of Chapter I, the initial conditions and the boundary condition for $y=0$ being zero for all φ_n, with the exception of $\varphi_1 = \int_{f(r)}^{t} P_0 dt$, $t - f(r)$, $r = R(t)$.

Consider the solution φ_n near the wave front $u = \tau_1$, $(t = \tau_1)$ of the homogeneous problem. If, for the region near the front AB of the homogeneous problem at which $u = \tau_1$, we write

$$P_0 = f_1(x, y) H(u - \tau_1) + f_2(x, y)(u - \tau_1), \tag{9.21}$$

where f_1 and f_2 are found in /27/ by the ray technique, then

$$\varphi_1 = f_1(x, y)(u - \tau_1) + f_2(x, y) \frac{(u - \tau_1)^2}{2}; \tag{9.22}$$

$H(x)$ is a unit step function. Suppose φ_n is represented near $u = \tau_1$ by

$$\varphi_n = \psi_{1,n}(x, y)(u - \tau_1) + \psi_{2,n} \frac{(u - \tau_1)^2}{2}, \tag{9.23}$$

where $\psi_{1,n}$ and $\psi_{2,n}$ are the values of the n-th term of a power series in k for the pressure P on the front in an inhomogeneous fluid. When (9.23) is substituted in (9.20), the right-hand side of (9.20) takes the form

$$\frac{f_n}{-a_0^2} = -2 \sum_1^{n-1} \left(\frac{\partial t_i}{\partial x} \frac{\partial \tau_1}{\partial x} + \frac{\partial t_i}{\partial y} \frac{\partial \tau_1}{\partial y} \right) \psi_{1,n-i} \delta(u - \tau_1) -$$

$$- \sum_1^{n-2} \psi_{1,n-i} \sum_1^{i-1} \left(\frac{\partial t_{i1}}{\partial x} \frac{\partial t_{i-i}}{\partial x} + \frac{\partial t_{i_1}}{\partial y} \frac{\partial t_{i-i_1}}{\partial y} \right) \delta(u - \tau_1) +$$

$$+ a_1 y \psi_{1,n-1} \delta(u - \tau_1) + 2 \sum_1^{n-1} \left(\frac{\partial t_i}{\partial x} \frac{\partial \psi_{1,n-i}}{\partial x} + \right.$$

$$\left. + \frac{\partial t_i}{\partial y} \frac{\partial \psi_{1,n-i}}{\partial y} - \frac{\partial t_i}{\partial x} \frac{\partial \tau_1}{\partial x} \psi_{2,n-i} - \frac{\partial t_i}{\partial y} \frac{\partial \tau_1}{\partial y} \psi_{2,n-i} \right) H -$$

$$- \sum_1^{n-2} \psi_{2,n-i} \sum_1^{i-1} \left(\frac{\partial t_{i_1}}{\partial x} \frac{\partial t_{i-i_1}}{\partial x} + \frac{\partial t_{i_1}}{\partial y} \frac{\partial t_{i-i_1}}{\partial y} \right) H +$$

$$+ a_1 y \psi_{2,n-1} H - \sum_1^{n-1} \psi_{1,n-i} \left(\frac{\partial^2 t_i}{\partial x^2} + \frac{\partial^2 t_i}{\partial y^2} \right) H. \tag{9.24}$$

where $\delta(x) = H'(x)$. The solution φ_n of (9.20) is found using (4.51) of Chapter I, and replacing f by f_n. For $u = \tau_1$, the terms containing $\delta(u - \tau_1)$ in (9.24) and corresponding to the integral over U in §4 of Chapter I will give a nonzero expression for φ_n. Since we require that $\varphi_n = 0$ on the

front in the inhomogeneous problem, the indicated term is set equal to zero, so that

$$-2\left(\frac{\partial t_1}{\partial x}\frac{\partial \tau_1}{\partial x}+\frac{\partial t_1}{\partial y}\frac{\partial \tau_1}{\partial y}-a_1 y\right)\psi_{1,\,n-1}-2\sum_{2}^{n-1}\left(\frac{\partial t_i}{\partial x}\frac{\partial \tau_1}{\partial x}+\frac{\partial t_i}{\partial y}\frac{\partial \tau_1}{\partial y}\right)\psi_{1,\,n-i}-$$

$$-\sum_{2}^{n-1}\psi_{1,\,n-i}\sum_{1}^{i-1}\left(\frac{\partial t_{i_1}}{\partial x}\frac{\partial t_{i-i_1}}{\partial x}+\frac{\partial t_{i_1}}{\partial y}\frac{\partial t_{i-i_1}}{\partial y}\right)=0. \tag{9.25}$$

In the first approximation

$$\frac{\partial t_1}{\partial x}\frac{\partial \tau_1}{\partial x}+\frac{\partial t_1}{\partial y}\frac{\partial \tau_1}{\partial y}-a_1 y=0; \tag{9.26}$$

the remaining part of (9.25) then gives

$$2\left(\frac{\partial t_i}{\partial x}\frac{\partial \tau_1}{\partial x}+\frac{\partial t_i}{\partial y}\frac{\partial \tau_1}{\partial y}\right)+\sum_{1}^{i-1}\left(\frac{\partial t_{i_1}}{\partial x}\frac{\partial t_{i-i_1}}{\partial x}+\frac{\partial t_{i_1}}{\partial y}\frac{\partial t_{i-i_1}}{\partial y}\right)=0. \tag{9.27}$$

We show that, according to (9.26) and (9.27), $t-u$ in (9.17) is the correction to the homogeneous equation of the front $t=\tau_1(x, y)$ in the inhomogeneous problem. If, in the wave front equation

$$\left(\frac{\partial t}{\partial x}\right)^2+\left(\frac{\partial t}{\partial y}\right)^2=\frac{1}{a^2(y)},$$

we substitute (9.17) in the form $t=\tau_1+\sum_{1}^{\infty}k^n t_n(x, y)$, then to zeroth order we obtain

$$\left(\frac{\partial \tau_1}{\partial x}\right)^2+\left(\frac{\partial \tau_1}{\partial y}\right)^2=\frac{1}{a_0^2}, \tag{9.28}$$

and to first order in k we obtain (9.26); equating terms in k^i, (9.27) is derived. Thus, if the difference $t-\tau_1$ between the equations of the inhomogeneous and homogeneous waves is given by $\sum_{1}^{\infty}k^n t_n(x, y)$ in (9.17), the potentials φ_n vanish on the front of the inhomogeneous wave and the solution in the form (4.51) of Chapter I will be valid up to the wave front.

Let us find $\frac{\partial \varphi_n}{\partial u}$, i.e., the pressure terms in k^n on the wave front. The obtained quadratures are such that as a result of the choice of t_n in f from (9.24) terms with $\psi_{2,\,n-i}$ cancel; hence, for $u=\tau_1$ we find (as for §4, Chapter I)

$$\frac{\partial \varphi_n}{\partial u}=\frac{1}{2}\int_{0}^{\frac{y}{\sin\varphi_0}}\Phi\sqrt{\frac{R_0(r)}{R_0(0)}}\,dr, \tag{9.29}$$

where

$$\Phi=2\sum_{1}^{n-1}\left(\frac{\partial t_i}{\partial x}\frac{\partial \psi_{1,\,n-i}}{\partial x}+\frac{\partial t_i}{\partial y}\frac{\partial \psi_{1,\,n-i}}{\partial y}\right)+$$

$$+\sum_{1}^{n-1}\psi_{1,\,n-i}\left(\frac{\partial^2 t_i}{\partial x^2}+\frac{\partial^2 t_i}{\partial y^2}\right),\quad \cos\varphi_0=\frac{a_0}{R'(\tilde{t}_0)},$$

and $\tilde{t}_0$. $R_0(r)$ have the same meaning as in §4 of Chapter I. Formula (9.29) for the pressure $\psi_{1,n} = \frac{\partial \varphi_n}{\partial u}$ on the wave front has a recurrent character, and is obtained using a ray representation. Along AB the pressure (/27/, p. 120) in the inhomogeneous case is given by

$$2\frac{\partial t}{\partial x}\frac{\partial P}{\partial x} + 2\frac{\partial t}{\partial y}\frac{\partial P}{\partial y} + P\left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2}\right) = 0.$$

In terms of the expansions

$$t = \tau_1 + \sum_1^{\infty} k^n t_n(x, y), \quad P = \sum_1^{\infty} k^{n-1} \psi_{1,n}$$

we have

$$\left(2\frac{\partial \tau_1}{\partial x}\frac{\partial \psi_{1,n}}{\partial x} + 2\frac{\partial \tau_1}{\partial y}\frac{\partial \psi_{1,n}}{\partial y}\right) + 2\sum_1^{n-1}\left(\frac{\partial t_i}{\partial x}\frac{\partial \psi_{1,n-i}}{\partial x} + \frac{\partial t_i}{\partial y}\frac{\partial \psi_{1,n-i}}{\partial y}\right) +$$

$$+ \sum_1^{n-1} \psi_{1,n-i}\left(\frac{\partial^2 t_i}{\partial x^2} + \frac{\partial^2 t_i}{\partial y^2}\right) + \psi_{1,n}\left(\frac{\partial^2 \tau_1}{\partial x^2} + \frac{\partial^2 \tau_1}{\partial y^2}\right) = 0. \qquad (9.30)$$

The first term in (9.30) can be written in the form

$$-2\frac{1}{a_0^2}\frac{d\psi_{1,n}}{dt'},$$

where $\frac{d}{dt'}$ denotes differentiation along a ray of the homogeneous problem. Hence /27/

$$\frac{\partial^2 \tau_1}{\partial x^2} + \frac{\partial^2 \tau_1}{\partial y^2} = -\frac{1}{a_0^2}\frac{1}{t' - \tilde{t}_0 - \frac{R'^3 \cos^2 \alpha_1}{a_0^2 R''}},$$

where $\tau_1 = t$ is the equation of the wave AB (Figure 8) for the homogeneous case, $\sin \alpha_1 = \frac{a_0}{R'(t_0)}$, $t' = \tilde{t}_0$ is the instant at which AB passes through the surface $y = 0$ along the ray MC; at the point $C, x = R(\tilde{t})$, $y = 0$. Multiplying (9.30) by $\frac{1}{2}\sqrt{t' - \tilde{t}_0 - \frac{R'^3 \cos^2 \alpha_1}{a_0^2 R''}}$ and integrating, we obtain

$$\psi_{1,n} = \frac{1}{2}\int_{\tilde{t}_0}^{t} \sum_1^{n-1} \psi_{1,n-i}\left(\frac{\partial^2 t_i}{\partial x^2} + \frac{\partial^2 t_i}{\partial y^2}\right) a_0^2 \sqrt{\frac{R_0(r)}{R_0(0)}}\, dt' +$$

$$+ \int_{\tilde{t}_0}^{t} \sum_1^{n-1}\left(\frac{\partial t_i}{\partial x}\frac{\partial \psi_{1,n-i}}{\partial x} + \frac{\partial t_i}{\partial y}\frac{\partial \psi_{1,n-i}}{\partial y}\right) a_0^2 \sqrt{\frac{R_0(r)}{R_0(0)}}\, dt',$$

where

$$R_0(r) = a_0\left(t - \tilde{t}_0 - \frac{r}{a_0} - \frac{R'^3 \cos^2 \alpha_1}{a_0^2 R''}\right),$$

which agrees with (9.29). The representation (9.29), however, is more convenient since it was obtained by a limiting process from the general solution in the whole region; this general solution, written in the form of quadratures, can be written in the axisymmetric case in terms of comparatively simple formulas of the type (9.16). To obtain an axisymmetric solution near the wave front, it is sufficient to multiply the function f_n in (9.24) by $\sqrt{x}$ and multiply expression (9.29) by $\sqrt{\frac{x - r\cos\varphi_0}{x}}$. As regards the convergence of the series (9.17), it seems that with sufficient information about the homogeneous solution $P_0(u, x, y)$ one can determine bounds for the right-hand sides of the equations for f_n, and then apply the Schwarz inequality to find limits for φ_n. These limits involve complicated expressions, but their derivation appears to be fairly simple and straightforward for each boundary condition. Since the f_n are analytic, the convergence is proved by the usual method (Babich, Voprosy dinamicheskoi teorii (Problems in Dynamics), Vol. 5, Leningrad. 1961). Far from the wave, relations of the type (9.16) /27/ give

$$P_n \approx y^{2n} a_1^n \int\int\int_{(n)} \frac{\partial^{2n} P_0}{\partial t^{2n}}\, dy dy dy$$

and the proof is simplified.

To obtain more effective expressions for the axisymmetric case we combine the method of /27/ with that described above. In the equation

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{1}{x}\frac{\partial P}{\partial x} = \left(\frac{1}{a_0^2} + a_1 k y\right)\frac{\partial^2 P}{\partial t^2}$$

we express t and P in the form (9.17) and derive the following equation for P_2:

$$\frac{\partial^2 P_2}{\partial x^2} + \frac{\partial^2 P_2}{\partial y^2} + \frac{1}{x}\frac{\partial P_2}{\partial x} - \frac{1}{a_0^2}\frac{\partial^2 P_2}{\partial u^2} =$$

$$= 2\frac{\partial^2 P_1}{\partial u \partial y}\frac{\partial t_1}{\partial y} + 2\frac{\partial^2 P_1}{\partial u \partial x}\frac{\partial t_1}{\partial x} + \frac{\partial P_1}{\partial u}\left(\frac{\partial^2 t_1}{\partial x^2} + \frac{\partial^2 t_1}{\partial y^2}\right) + \frac{1}{x}\frac{\partial P_1}{\partial x}\frac{\partial t_1}{\partial x} + a_1 y \frac{\partial^2 P_1}{\partial u^2}.$$

For constant $R'(t) = V$, equation (9.3) gives $t_1 = \frac{a_1 y^2}{4\sqrt{1/a_0^2 - 1/V^2}}$ to first order and the right-hand side of the equation for P_2 is

$$2\frac{\partial^2 P_1}{\partial u \partial y}\frac{a_1 y}{2\sqrt{\frac{1}{a_0^2} - \frac{1}{V^2}}} + \frac{\partial P_1}{\partial u}\frac{a_1}{2\sqrt{\frac{1}{a_0^2} - \frac{1}{V^2}}} + a_1 y \frac{\partial^2 P_1}{\partial u^2}.$$

The solution for P_1 in the homogeneous case is known /27/. The Laplace transforms of P_1 and P_2 are written (as in /27/) in the form

$$\overline{P}_1 = \int_0^\infty A(\lambda) \cdot e^{-\sqrt{\frac{s^2}{a_0^2} + \lambda^2}\, y} I_0(\lambda x)\, d\lambda,$$

$$\overline{P}_2 = \int_0^\infty A(\lambda) I(y) I_0(\lambda x)\, d\lambda,$$

corresponding to which the equation for P_2 becomes

$$I''(y) - \left(\lambda^2 + \frac{s^2}{a_0^2}\right) I(y) \quad \left(- \frac{a_1 y}{\sqrt{\frac{1}{a_0^2} - \frac{1}{V^2}}} s \sqrt{\frac{s^2}{a_0^2} + \lambda^2} + \right.$$

$$\left. + \frac{a_1 s}{2\sqrt{\frac{1}{a_0^2} - \frac{1}{V^2}}} + a_1 y s^2 \right) e^{-\sqrt{\frac{s^2}{a_0^2} + \lambda^2}\, y};$$

a solution subject to the boundary conditions $I(0) = 0$, $I(\infty) = 0$ is obtained in the form

$$I(y) = (C_0 y + C_1 y^2)\, e^{-\sqrt{\frac{s^2}{a_0^2} + \lambda^2}\, y}$$

$$C_0 = - \frac{a_1 s^2}{4\left(\lambda^2 + \frac{s^2}{a_0^2}\right)}, \quad C_1 = \frac{a_1 s}{4\sqrt{\frac{1}{a_0^2} - \frac{1}{V^2}}} - \frac{a_1 s^2}{4\sqrt{\lambda^2 + \frac{s^2}{a_0^2}}}.$$

Transforming back from $\bar{P}_2$ to P_2 we obtain

$$P_2 = \frac{a_1}{4} y^2 \int_u^\infty \frac{\partial^2 P_1}{\partial u^2}\, dy - \frac{a_1 y}{4} \int_u^\infty \int_u^y \frac{\partial^2 P_1}{\partial u^2}\, dy dy - \frac{a_1 y^2}{4\sqrt{\frac{1}{a_0^2} - \frac{1}{V^2}}} \frac{\partial P_1}{\partial u};$$

the first term, which has a singularity near the front

$$u = y\sqrt{\frac{1}{a_0^2} - \frac{1}{V^2}} + \frac{x}{V},$$

and where P_1 is given by (9.21), cancels with the third term. In the two-dimensional problem

$$P_2 = - \frac{a_1}{4} y \frac{1}{\frac{1}{a_0^2} - \frac{1}{V^2}} f_1$$

on the front; this agrees with the solution of the inhomogeneous problem to first order. Such simple results cannot be obtained everywhere in the region for arbitrary $R(t)$.

APPENDIX I

The solution obtained in §6 of Chapter II is valid for $\frac{y_1}{r} \ll 1$. Suppose the boundary condition at the fluid surface is

hence

$$k \ln \frac{r}{a_0 y_1} \sim \frac{y_2 - y_1}{F(y_2) - F(y_1)}, \left| \ln \frac{y_2}{y_1} \right| \ll \left| \ln \frac{y_1}{r} \right|.$$

Differentiating this relation along the shock wave with respect to r and eliminating r and dr from both it and formula (1.2), we have

$$\int_{y_1}^{y_2} F(y)\, dy = \frac{1}{2}(y_2 - y_1)(F(y_1) + F(y_2)). \tag{1.3}$$

Condition (1.3) implies an equality of the areas between $F(y)$ and the secant /26/ (Figure 14').

In Figure 14' the segment BC corresponds to y_1, the region ahead of the discontinuity, and the segment CD corresponds to y_2, the region behind the discontinuity. A tail shock forms at the point of inflection on CD. As $t \to \infty$, we have $y_1 \to y_0$, $y_2 \to \infty$ in the discontinuity equation (1.3), so that

$$k \ln \frac{r}{a_0 y_0} = \frac{y_2 - y_1}{-F(y_1)},$$

$$\int_{y_0}^{\infty} F(y)\, dy = \frac{1}{2}(y_2 - y_1) F(y_1)$$

or

$$-F(y_1) = \left(\frac{-2 \int_{y_0}^{\infty} F(y')\, dy'}{k \ln \frac{r}{a_0 y_0}} \right)^{\frac{1}{2}}. \tag{1.4}$$

The equation of the tail shock is

$$t - \frac{r}{a_0} = y_0 + \sqrt{k \ln \frac{r}{a_0 y_0}} \sqrt{-2 \int_{y_0}^{\infty} F(y')\, dy'}. \tag{1.5}$$

In virtue of

$$\int_0^{\infty} F(y')\, dy' = 0 \tag{1.6}$$

the coefficients of $\pm \sqrt{k \ln \frac{r}{a_0 y_0}}$ in equations (1.5) and (6.19) of Chapter II, are equal.

From (6.9) of Chapter II, the pressure behind the tail shock is

$$P = \frac{F(y_2)}{r},$$

$$R = R^\circ(t),\ R^{\circ 2}(t) - D_1^2 \frac{(t+t_0)^2}{\left(1+\frac{t}{t_2}+\frac{t_0}{t_2}\right)^2} - D_1^2 \frac{t_0^2}{\left(1+\frac{t_0}{t_2}\right)^2},$$

$$P_\phi = P_1 \frac{t_0}{t_0+t}. \qquad (1.1)$$

where t_0, t_2 and D are constant, $R(\infty)$ is finite, and $R'(0) = \infty$. Figure 14' then shows $F(y_1)$ and P at the point B of the shock wave on $0y$; the curve $A'C'D'$ represents $F(y_1)$ for the conditions (1.1), and the curve $OABCD$ represents $F(y_1)$ for finite $R'(0)$. The function $F(y_1)$ is represented away from the $0y$ axis by the curve $GA'C'D'$. Note that (6.5) of Chapter II is satisfied at all times for a finite $R'(0)$, since for small y_1 the function $F(y_1)$ is small and the theory holds; under conditions (1.1) for small y_1, however, (6.4) of Chapter II gives an infinite value of P at $t = 0$, since $F(0) \neq 0$. Therefore, the problem must be solved separately for finite t and large t and the solutions matched. This is carried out on Figure 14' for the $0y$ axis in Tables 2 and 1 respectively; the last column in Table 1 is calculated by formula (1.7) for the tail shock.

The solution (6.9), (6.14) of Chapter II is based on the linear solution (2.2) of Chapter II for the region $r_1 < a_0 t$.

In the integrand of solution (6.10) we have the expression $y_1 + \frac{1}{a_0} r\cos\theta\cos\psi > 0$, since $y_1 \approx y_0$ always for large t, and $y_1 > 0$. Since the region on the surface at which pressure is applied is limited to $0 < r < R(\infty)$, the pressure on the shock wave AB is given by the one-dimensional theory along the rays (6.10), where (2.2) does not need to be replaced by (2.4) in the expression for $F(y_1)$. When $y_1 < 0$, one should use (2.4) and take $y_2 < y_1 < y_0$ as the integration limits in (6.14), as pointed out in §6 of Chapter II (Figure 14'). When $R(t) = At^\gamma$, then $F(y_1) \sim y_1^{2\gamma-2}$, and from (6.5) $y_1 \sim \frac{1}{(\ln y)^{\frac{2}{2\gamma-3}}}$.

In §6 of Chapter II the attenuation of a shock wave for large t was determined. The conditions for the formation of a tail shock were found and it was shown that it always forms in the given problem.

Let us now find the tail shock. Suppose the characteristics (6.2) of Chapter II ahead of and behind the tail shock correspond to the parameters y_1, y_2. Then, from Pfreim's formula along the tail shock, we have

$$\frac{\partial}{\partial r}\left(y_1 - kF(y_1)\ln\frac{r}{a_0 y_1}\right) = \frac{1}{2}\left(-F(y_1)\frac{k}{r} - F(y_2)\frac{k}{r}\right), \qquad (1.2)$$

and along the shock wave

$$y_1 - kF(y_1)\ln\frac{r}{a_0 y_1} = y_2 - kF(y_2)\ln\frac{r}{a_0 y_2},$$

$$k = \frac{1}{a_0 B n}\frac{n+1}{2};$$

where, for example,

$$F(y_2) = \frac{R^2(\infty)}{2a_0} \frac{\partial P_\Phi(y_2)}{\partial y_2}$$

on the axis of symmetry. Suppose $P_\Phi(t) = P_1 \frac{t_0}{t_0 + t}$; then for large y_2

$$F(y_2) = -\frac{R^2(\infty)}{2a_0} P_1 \frac{1}{\left(1 + \frac{y_2}{t_0}\right)^2} \frac{1}{t_0}.$$

According to (1.4)

$$-F(y_1) = \left(\frac{-2\int_{y_0}^{\infty} F(y')\, dy'}{k \ln \frac{r}{a_0 y_0}} \right)^{\frac{1}{2}},$$

and so from (1.3)

$$y_2 = \left(k \ln \frac{r}{a_0 y_0}\right)^{\frac{1}{2}} \left(-2 \int_{y_0}^{\infty} F(y')\, dy' \right)^{\frac{1}{2}};$$

hence

$$P = -\frac{R^2(\infty)}{2a_0} \frac{P_1}{r} t_0 \frac{1}{k \ln \frac{r}{a_0 y_0} \, 2 \int_0^{y_2} F(y)\, dy}. \tag{1.7}$$

APPENDIX II

The mathematical formulation of the problem in Chapter I agrees with that of the problem on conical flows in the case of flow at a small angle of attack α' around the tip of a triangular wing and the edge of a rectangular wing. The disturbed region for a triangular wing with supersonic edges is shown in Figure 5, and for a tetragonal wing in Figure 18 /3/. The solution on the arc MN for the flow around a rectangular wing is given by Lighthill /3/ in the form

$$v_r \sim \frac{3n(n+1) \frac{V^6}{a_0^6}}{4\pi^2 \left(\frac{V^2}{a_0^2} - 1\right)^2} \alpha'^2 \frac{\sin^2 \theta}{\cos^2 \theta \sin^2 \frac{\theta}{2}}, \tag{2.1}$$

where n is the adiabatic exponent, V is the velocity of the incoming flow, a is the velocity of sound, α' is the angle of attack, and the $0z$ axis is

directed along the incoming flow; polar coordinates $r, \theta\ (x = r\cos\theta,\ y = r\sin\theta)$ are introduced in a plane normal to $0z$ (Figure 18).

Near the point B (Figure 18) $\theta - \frac{3\pi}{2} \sim \gamma^{\alpha'}$, $\gamma = \frac{(v_r)_B}{a_0}$, and using the treatment in § 3 of Chapter II it is found that formula (2.1) is correct for $\alpha' < \frac{1}{2}$; for $\alpha' = \frac{1}{2}$ the solution is a two-dimensional short wave (see below) shown in Figure 6. The solution for this case /8/ appears to be incorrect, since the conclusion that there is a discontinuity near the point B (Figure 18) in the form of a jump in $v_y = W$ from the value $V\alpha'$ in front of the point B to $\frac{8}{5} V\alpha'$ behind it, contradicts the existence there of a rarefaction (from B to N). The reason for this is that in the solution of /8/ the derivatives of the parameters with respect to r, and the parameters v_r, v_z themselves, are assumed to be first-order small quantities in γ, whereas we showed that the derivatives with respect to r are finite there (see § 4, Chapter I).

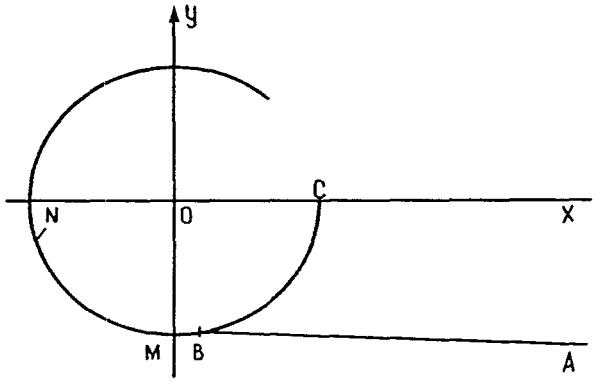


FIGURE 18.

The coordinates of B can be found by determining the point of intersection of the straight discontinuity AB (Figure 18), given by

$$\frac{r\sin\theta}{z} = \frac{1}{\sqrt{\frac{V^2}{a_0^2} - 1}} + \frac{1}{\left(\frac{V^2}{a_0^2} - 1\right)^2} \frac{V^3}{a_0^3} \frac{n+1}{4} \frac{(v_r)_B}{a_0},$$

with the Mach cone BC, given by

$$\frac{r}{z} = \frac{1}{\sqrt{\frac{V^2}{a_0^2} - 1}} + \frac{1}{\left(\frac{V^2}{a_0^2} - 1\right)^2} \frac{V^3}{a_0^3} \frac{n+1}{2} \frac{(v_r)_B}{a_0},$$

where $v_r = V\alpha'$ at B in the region of constant flow behind AB.

For a triangular wing with opening semiangle β, around which there is flow with velocity V in the direction of the $0z$ axis at an angle of attack α',

the following equations apply for steady-state motion:

$$v_r \frac{\partial v_r}{\partial r} + v_\theta \frac{\partial v_r}{r\partial\theta} + v_z' \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r}, \tag{2.2}$$

$$v_r \frac{\partial v_\theta}{\partial r} + v_\theta \frac{\partial v_\theta}{r\partial\theta} + v_z' \cdot \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} = -\frac{1}{\rho}\frac{\partial P}{r\partial\theta}, \tag{2.3}$$

$$v_r \frac{\partial v_z'}{\partial r} + v_\theta \frac{\partial v_z'}{r\partial\theta} + v_z' \frac{\partial v_z'}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z}, \tag{2.4}$$

$$\frac{\partial}{\partial r}(r\rho v_r) + \frac{\partial}{\partial\theta}(\rho v_\theta) + \frac{\partial (r\rho v_z')}{\partial z} = 0, \tag{2.5}$$

where polar coordinates are introduced in the (x, y)-plane and

$$v_r = \frac{\partial\varphi}{\partial r},\ v_z' = V + \frac{\partial\varphi}{\partial z},\ v_\theta = \frac{\partial\varphi}{r\partial\theta},\ v_z = \frac{\partial\varphi}{\partial z}.$$

In a plane perpendicular to the $0z$ axis we have the flow delineated in Figure 5. There is a constant flow region in ABC, and in the region $C_1B_1B_1'C_1'$ (the linear problem) the boundary-value problem for v_y is identical to that for P in Chapter I. Near B_1B_1' (according to (1.9) of Chapter I in which $\frac{V}{a_0}$ is replaced by $\operatorname{tg}\beta\sqrt{\frac{V^2}{a_0^2}-1}$), we have

$$\frac{v_y}{V\alpha'} = f_2(\theta)\sqrt{\frac{z - r\sqrt{\frac{V^2}{a_0^2}-1}}{z}}, \tag{2.6}$$

where

$$f_2(\theta) = \frac{2\sqrt{2}}{\pi}\frac{\sin\theta\operatorname{tg}\beta\sqrt{\frac{V^2}{a_0^2}-1}}{1-\operatorname{tg}^2\beta\left(\frac{V^2}{a_0^2}-1\right)\cos^2\theta}.$$

Near the point B (Figure 5) $f_2(\theta) = \frac{\sqrt{2}}{\pi}\frac{1}{\theta-\theta_0}$, where θ_0 is the polar angle of the point B_1 (Figure 5). In virtue of the conicity near BB' in the linear case /26/, the solution

$$v_r = -\sqrt{\frac{V^2}{a_0^2}-1}\,v_z,\ v_z = -A(\theta)\sqrt{\frac{z - r\sqrt{\frac{V^2}{a_0^2}-1}}{r\sqrt{\frac{V^2}{a_0^2}-1}}} \tag{2.7}$$

is obtained, where

$$A(\theta) = \frac{f_2(\theta)\operatorname{tg}\beta\, V\alpha'}{\sqrt{\operatorname{tg}^2\beta\left(\frac{V^2}{a_0^2}-1\right)-1}},\ A > 0$$

and near B

$$v_y = v_r \sin\theta.$$

To obtain the pressure along BB' the linear characteristics are replaced by improved ones. Replacing $z - r\sqrt{\frac{V^2}{a_0^2} - 1}$ by y_1, where $y_1 = \text{const}$ is the equation of the characteristics, and integrating the characteristics

$$z - r\sqrt{\frac{V^2}{a_0^2} - 1} - z'_r \frac{\partial z'}{\partial r} = -\frac{n+1}{2}\frac{V^3}{a_0^4}\frac{v_r}{\frac{V^2}{a_0^2} - 1},$$

we obtain $v_r = -A(\theta)\frac{\sqrt{y_1}}{\sqrt{\frac{V^2}{a_0^2} - 1}\sqrt{r}}$; $\sqrt{y_1}$ has the form

$$\sqrt{y_1} = \frac{q(r) + \sqrt{q^2(r) + 4\left(z - r\sqrt{\frac{V^2}{a_0^2} - 1}\right)}}{2},$$

where

$$q(r) = (n+1)\frac{V^3}{a_0^4}\sqrt{r}\; A(\theta)\frac{1}{\left(\frac{V^2}{a_0^2} - 1\right)^{3/4}}.$$

If $\sqrt{y_1}$ is eliminated, the solution that Lighthill found by another technique is obtained in the neighborhood of BB'. On the shock front BB'

$$v_r = \frac{3}{4}A^2(\theta)\frac{V^3}{a_0^4}(n+1)\frac{1}{\left(\frac{V^2}{a_0^2} - 1\right)^{1/2}},$$

where $A(\theta)$ is given just after (2.7).

Whitham /43/* first showed the possibility of obtaining this solution on the shock front; the solution itself was discovered by Lighthill using Whitham's method /4/.

The solution is determined by the two variables $\frac{r}{z}$ and θ, and satisfies equations (2.2)—(2.5) (which are one-dimensional in r/z) at finite distances from the point B on the arc BB'. At distances from the point B such that $\theta - \theta_0 \sim \sqrt{\gamma}$, where $\gamma = v_{r_0}/a_0$ and v_{r_0} is the radial velocity at B,

$$\cos\theta_0 = \frac{1}{\operatorname{tg}\frac{\alpha'}{2}\sqrt{V^2 a_0^2 - 1}}, \quad v_{r_0} = \frac{V\alpha'}{\sin\theta_0},$$

the solution becomes two-dimensional, since the approximations in § 4 of Chapter I are valid, as follows from the solution along the arc BB' in the

* The paper also contains a number of results (§ 6, Chapter II) for an explosion, and the theory for a wing with a rounded leading edge.

form

$$v_r \sim \frac{3}{4}(n+1)^2 \frac{2}{\pi^2} \frac{1}{(\theta-\theta_0)^2} \frac{\operatorname{tg}^2\beta \dfrac{V^5}{a_0^4} \alpha'^2}{\left(\operatorname{tg}^2\beta\left(\dfrac{V^2}{a_0^2}-1\right)-1\right)\left(\dfrac{V^2}{a_0^2}-1\right)^{1/2}}. \tag{2.8}$$

Consider now the equations of two-dimensional short waves, which hold at points of the arc BB' such that $\theta - \theta_0 \sim \sqrt{\gamma}$.

The equation of the cone of initial disturbances CB (Figure 5) becomes

$$\frac{r}{z} = \operatorname{tg}\alpha, \tag{2.9}$$

where $\sin\alpha = \dfrac{a + \dfrac{v_r}{\sqrt{1-a_0^2/V^2}}}{V}$ and a is the velocity of sound. Bernoulli's equation yields

$$\frac{a^2}{n-1} + \frac{v_z'^2}{2} + \frac{v_r^2}{2} + \frac{v_\theta^2}{2} = \frac{a_0^2}{n-1} + \frac{V^2}{2}. \tag{2.10}$$

where n is the adiabatic exponent, $v_z = V + v_z$, and v_r, v_θ, v_z are small. If we retain first-order small quantities, (2.10) becomes

$$\frac{a^2}{n-1} + V v_z = \frac{a_0^2}{n-1}. \tag{2.11}$$

When system (2.4)—(2.7) is expressed in terms of the variables

$$\xi = \frac{r\sqrt{\dfrac{V^2}{a_0^2}-1}}{z}, \quad \xi = 1 + M_0\delta, \quad \theta - \theta_0 = \sqrt{M_0}\, Y,$$

$$v_\theta = M_0^{\frac{3}{2}} a_0 \nu, \quad v_r = a_0 M_0 \mu, \tag{2.12}$$

$$M_0 = \gamma, \tag{2.13}$$

we obtain

$$(n+1)\lambda\mu\frac{\partial\mu}{\partial\delta} - 2\delta\frac{\partial\mu}{\partial\delta} + \frac{\partial\nu}{\partial Y} + \mu = 0,$$

$$\frac{\partial\mu}{\partial Y} = \frac{\partial\upsilon}{\partial\delta}, \qquad \lambda = \frac{V^3}{a_0^3\left(\dfrac{V^2}{a_0^2}-1\right)^{1/2}}, \tag{2.14}$$

$$\sqrt{\frac{V^2}{a_0^2}-1}\, v_z = -v_r \tag{2.15}$$

to first order. The equation of the cone C_1B_1 in the linear case has the form

$$\frac{r}{z} = \frac{1}{\sqrt{\dfrac{V^2}{a_0^2}-1}}, \quad \sin\alpha_0 = \frac{a_0}{V}. \tag{2.16}$$

From (2.9) the cone CB becomes in the first approximation

$$\frac{r}{z} = \frac{1}{\sqrt{\dfrac{V^2}{a_0^2}-1}} + \frac{n+1}{2}\frac{1}{\sqrt{\dfrac{V^2}{a_0^2}-1}}\lambda\frac{(v_{r_1})_B}{a_0}. \tag{2.17}$$

Let us find the point B at which the shock wave AB intersects the cone BC. The equation of the shock wave AB in the linear approximation (§ 1, Chapter I)

is

$$\eta = \frac{\operatorname{tg}\beta - \xi'}{\sqrt{\left(\frac{V^2}{a_0^2} - 1\right)\operatorname{tg}^2\beta - 1}}, \quad \xi' = \frac{x}{z}, \quad \eta = \frac{y}{z}. \tag{2.18}$$

The angle of inclination to $0x$ of the shock wave AB_1 in the linear problem is given by

$$\sin\beta_0 = \frac{1}{\operatorname{tg}\beta\sqrt{\frac{V^2}{a_0^2} - 1}}. \tag{2.19}$$

Pfreim's formula gives the equation of the shock wave AB to first order as

$$\eta = \frac{\operatorname{tg}\beta - \xi'}{\sqrt{\dfrac{\operatorname{tg}^2\beta}{\left(\dfrac{r}{2z} + \dfrac{1}{2\sqrt{V^2/a_0^2 - 1}}\right)^2} - 1}}, \tag{2.20}$$

where $\frac{r}{z}$ is given in (2.17), or

$$\eta = \frac{\operatorname{tg}\beta - \xi'}{\sqrt{\left(\frac{V^2}{a_0^2} - 1\right)\operatorname{tg}^2\beta - 1}} + \frac{(\operatorname{tg}\beta - \xi')\frac{n+1}{4}\operatorname{tg}^2\beta\frac{V^3}{a_0^3}\frac{(v_{r_1})_B}{a_0}}{\sqrt{\frac{V^2}{a_0^2} - 1}\left(\left(\frac{V^2}{a_0^2} - 1\right)\operatorname{tg}^2\beta - 1\right)^{\frac{3}{2}}}. \tag{2.21}$$

The intersection of (2.17) with (2.21) defines the point B:

$$\xi' - \frac{1}{\operatorname{tg}\beta\left(\frac{V^2}{a_0^2} - 1\right)} = \sqrt{\frac{\left(\frac{V^2}{a_0^2} - 1\right)\operatorname{tg}^2\beta - 1}{\operatorname{tg}^2\beta\left(\frac{V^2}{a_0^2} - 1\right)}\frac{n+1}{2}\lambda}. \tag{2.22}$$

where

$$\cos\theta_0 = \sin\beta_0. \tag{2.23}$$

Since $\xi' \approx \frac{1}{\sqrt{V^2/a_0^2 - 1}}\cos\theta$ at the point B, (2.22) yields

$$\theta_0 - \theta = \sqrt{\frac{n+1}{2}\lambda\frac{(v_{r_1})_B}{a_0}}. \tag{2.24}$$

The relation for the tangential velocity component at the front gives the following expression for v_θ:

$$v_\theta = v_{r_1}(\theta_0 - \theta), \tag{2.25}$$

or at the point B

$$v_\theta = v_{r_1}\sqrt{\frac{n+1}{2}\lambda\frac{v_{r_1}}{a_0}}. \tag{2.26}$$

(The treatment remains the same for a tetragonal wing, except that $\theta_0=\frac{3\pi}{2}$ in (2.24) and on the right-hand side a minus sign appears before the square root.) It then follows from (2.14) and (2.25) that one may set

$$v_\theta = M_0^{\frac{3}{2}} a_0 \sqrt{\lambda} \sqrt{\frac{n+1}{2}}\, v', \tag{2.27}$$

$$\theta - \theta_0 = \sqrt{\lambda} \sqrt{\frac{n+1}{2} M_0}\, Y', \tag{2.28}$$

$$\delta = \delta' \lambda \frac{n+1}{2}; \tag{2.29}$$

(2.14) agrees with (4.6) of Chapter I, where $v = v'$, $Y = Y'$, $\delta = \delta'$ and at the point B, $\mu = 1$, $v' = 1$, $\delta' = 1$, $Y' = -1$.

The solution $\mu(Y')$ along the shock wave is given in Figure 6.* The same results are obtained for the edge of a rectangular wing.

Thus, at the junction of the elliptic and hyperbolic regions on the shock wave, the solutions of nonstationary problems of the indicated type and of steady-state conical flows in dimensionless variables are given by the same formulas.

A short-wave representation in the vicinity of the point B in Figure 5 is given in /47/. If we take

$$\theta_0 = \frac{3\pi}{2},\ \delta' = -2\xi,\ \ Y' = \eta\sqrt{2},\ \varphi = Vz(1+f),$$

$$f = \frac{1}{\sqrt{V^2/a_0^2 - 1}}\, \alpha'^2 (n+1)\lambda \frac{V}{a_0} F(\xi, \eta),$$

the particular solution obtained there assumes the form

$$F = (\eta + \eta_0)^4 G(\zeta), \quad \zeta = \frac{\xi}{(\eta+\eta_0)^2}.$$

From the conditions satisfied by the velocity component $\frac{\partial\varphi}{\partial s}$ tangential to the shock and the velocity of the shock /47/

$$\frac{\partial F}{\partial \xi} = 4\xi + 2\left(\frac{d\xi}{d\eta}\right)^2,$$

it is found that using the coordinates of the point B (Figure 18) corresponding to which $\frac{\partial F}{\partial \xi} = -1$, $\xi = -\frac{1}{2}$, $\eta = \frac{1}{\sqrt{2}}$, then $G'(\zeta_0) = 2\zeta_0$, $\zeta_0 = -\frac{1}{4}$, $\eta_0 = \frac{1}{\sqrt{2}}$ at B /47/.

According to /47/, the equation of the shock in the vicinity of B is

$$\xi = -\frac{1}{4}\left(\eta + \frac{1}{\sqrt{2}}\right)^2,$$

* Figure 6 agrees qualitatively with the numerical computations of /58/ and /59/; unfortunately, they do not give the pressure distribution along the shock wave. In the solution (4.6') of Chapter I for Figure 6, $B = 0.5$ is chosen; in the general case, a short wave can be obtained by matching solutions (4.6') for different B.

whence its curvature at this point is $k_B = -\frac{1}{2}$. The following relations exist:

$$\frac{\partial F}{\partial \xi} = -\frac{1}{2}\left(\tau_1 + \frac{1}{\sqrt{2}}\right)^2 : 2\sqrt{\frac{v_r}{a^*}} - \gamma'' + 1;$$

$$2\sqrt{\frac{v_r}{a_0}\frac{n+1}{2}}\,\lambda = \theta - \theta_0 \qquad \theta_1 - \theta_0.$$

The equations of motion and continuity give /47/

$$(2\zeta - 4\zeta^2 - G')\,G'' - (1 + 10\zeta)\,G' + 12G = 0,$$

the initial condition being taken in the form

$$\zeta = -\frac{1}{4}, \quad G = \frac{1}{24} \quad G' = -\frac{1}{2}.$$

Now, $\nu' = \sqrt{2}\,\frac{\partial F}{\partial \tau_1}$ and therefore at the point B(Figure 18) $\nu' = -\frac{1}{3}$; this contradicts $\nu' = 1$.

Thus, the particular solution of /47/ cannot satisfy all the conditions at the point B: it agrees with formula (4.70) of Chapter I at the point B on BB' (Figure 5) where $\frac{\partial \mu}{\partial Y} = -1$.

Consider the flow around a triangular wing whose cross section z =const yields the contour $\eta = \eta(\xi)$ defined by the variables

$$\xi = \frac{x}{z}\,(M^2 - 1)^{\frac{1}{2}}, \qquad \eta = \frac{y}{z}\,(M^2 - 1)^{\frac{1}{2}}. \tag{2.30}$$

In the region $B_1C_1C_1'B_1'$ (Figure 5) for the linear case, the solution in terms of the variables

$$\xi = r'\cos\theta, \quad \eta = r'\sin\theta, \tag{2.31}$$

$$r' = \frac{2\varepsilon}{1+\varepsilon^2}, \quad z = \varepsilon e^{i\theta}, \quad k = \frac{2z}{1+z^2}, \quad k = \xi_2^0 - i\eta_2 \tag{2.32}$$

becomes (according to §1 of Chapter I)

$$v_y = \frac{1}{\pi}\,\eta_{i2} \int_{-\operatorname{tg}\beta\sqrt{(M^2-1)\eta_2}}^{\operatorname{tg}\beta\sqrt{(M^2-1)\eta_2}} \frac{(v_{y_1})_0\,d\xi_2}{(\xi_2 - \xi_2^0)^2 + \eta_{i2}^2}, \tag{2.33}$$

where $(v_{y_1})_0$ on the wing is found from the condition

$$(v_{y_1})_0 = V\frac{\partial y}{\partial z}. \tag{2.34}$$

$y(z, x)$ representing the contour $\frac{y}{z} = (M^2-1)^{-\frac{1}{2}} \eta(\xi)$. As an example consider the parabolic wing contour

$$(M^2-1)^{-\frac{1}{2}} \gamma_1(\xi) = \gamma \left(\operatorname{tg}\beta (M^2-1)^{\frac{1}{2}} - \xi\right)^2; \tag{2.35}$$

then

$$\frac{\partial y}{\partial z} = \gamma \left(\operatorname{tg}^2\beta (M^2-1) - \xi^2\right), \tag{2.36}$$

where γ is small.

In virtue of (2.32) for $y=0$, $\xi = \xi_2^0$ near C_1B_1, whose equation has the form

$$\frac{r}{z} = (M^2-1)^{-\frac{1}{2}}, \quad r = \sqrt{x^2+y^2}$$

where

$$r' = 1-\alpha, \quad \varepsilon = 1-\sqrt{2\alpha}, \quad \alpha = 1-(M^2-1)^{\frac{1}{2}} \frac{r}{z}.$$

After calculating v_y we obtain

$$\frac{v_y - \gamma\left(\operatorname{tg}^2\beta (M^2-1) - \frac{1}{\cos^2\theta}\right) V}{V\sqrt{2}\,\gamma \frac{\sin\theta}{\cos^2\theta}\frac{2}{\pi}} = A(\theta)\sqrt{\alpha}, \tag{2.37}$$

where

$$A(\theta) = -2\operatorname{tg}\beta (M^2-1)^{\frac{1}{2}} - \frac{1}{\cos\theta} \ln \frac{\operatorname{tg}\beta (M^2-1)^{\frac{1}{2}} - \frac{1}{\cos\theta}}{\operatorname{tg}\beta (M^2-1)^{\frac{1}{2}} + \frac{1}{\cos\theta}}; \tag{2.38}$$

$A(\theta)$ varies for different θ from zero to its value at the point B, and

$$\cos\theta = \frac{1}{\operatorname{tg}\beta (M^2-1)^{1/2}} \tag{2.39}$$

varies from some negative value at $\theta=0$ to a positive infinitely large value at B. In the region ABC (Figure 5) the solution can be found by the method of simple waves /7/. The equation of the surface, which intersects the (x, y)-plane along the characteristic C_2M (Figure 5), is /7/

$$x\sin\theta\cos\varphi + y\sin\theta\sin\varphi + z\cos\theta - (a + v_n + V\cos\theta)\frac{z}{V} = 0, \tag{2.40}$$

where the time would be replaced by $\frac{z}{V}$ in a steady-state motion;

$$v_n = v_x\sin\theta\cos\varphi + v_y\sin\theta\sin\varphi + v_z\cos\theta$$

is the particle velocity normal to the surface (2.40), θ is the angle between the normal to (2.40) and the $0z$ axis, where $\cos\theta = \frac{a + v_n}{V}$, and the angle φ is found from (2.40) at the point C_2 (Figure 5). If

$$\xi_1 = \xi', \quad \eta_1 = 0, \quad \xi' \cos\varphi = \frac{a + v_n}{V \sin\theta}. \tag{2.41}$$

then $a = a(\xi')$ and all the remaining parameters are constant on (2.40), where $\xi_1 = \frac{x}{z}$, $\eta_1 = \frac{y}{z}$. In the linear case equation (2.40) becomes

$$\xi_1 \cos\varphi + \eta_1 \sin\varphi = (M^2 - 1)^{-\frac{1}{2}}, \tag{2.42}$$

$$\cos\varphi = \frac{(M^2-1)^{-\frac{1}{2}}}{\xi'}, \quad \xi_1 \cos\varphi + \eta_1 \sin\varphi = \xi' \cos\varphi,$$

where all the parameters depend on ξ' and are constant along (2.42). Using (2.34) and (2.36), v_y is given by

$$v_y = V \cdot \gamma (M^2 - 1)(\mathrm{tg}^2\beta - \xi'^2). \tag{2.43}$$

Near BC, (2.42) implies that for $r = (\xi^2 + \eta^2)^{\frac{1}{2}}$,

$$r = (M^2 - 1)^{-\frac{1}{2}} + r_2, \quad r\cos(\varphi - \theta) = (M^2 - 1)^{-\frac{1}{2}}$$

where

$$\xi = r\cos\theta, \quad \eta = r\sin\theta$$

and

$$r_2 = (M^2 - 1)^{-\frac{1}{2}} \frac{(\theta - \varphi)^2}{2}, \quad \theta - \varphi = -\sqrt{\frac{2r_2}{(M^2-1)^{-\frac{1}{2}}}}. \tag{2.44}$$

Now, near BC in the region ABC we have, from (2.42) and (2.43),

$$\frac{v_y - \gamma\left(\mathrm{tg}^2\beta\,(M^2-1) - \frac{1}{\cos^2\theta}\right)V}{\gamma\,(M^2-1)\frac{\sin\theta}{\cos^2\theta}\sqrt{2}\,V} = -\left(\frac{1}{M^2-1}\right)^{\frac{3}{4}} \frac{1}{\cos\theta}\, 2\,(r'-1)^{\frac{1}{2}}. \tag{2.45}$$

Thus, the coefficient $B(\theta)$ of $(r'-1)^{\frac{1}{2}}$ in the present case is negative, and according to § 3 of Chapter I this results in a shock wave at BC /3/.

APPENDIX III

Let us find the attenuation of a shock wave in an inhomogeneous medium (§ 9, Chapter II).

For large times t or large distances y, the linear pressure distribution in a half-space is given by (5.1) of Chapter I (/27/, p. 78)., where in the

elementary solution for V only the first term V_0 can be retained, since the remaining terms will be of the order of $V_n \sim \left(\sqrt{\frac{\tau'}{y}}\right)^n$ and consequently small for large y. Hence, for large y and finite τ'

$$P = -\frac{1}{\pi}\frac{\partial}{\partial y}\int_0^{\sqrt{\psi - x^2}} d\eta \int_{-r_1^*}^{r_0^*} \frac{P_1 V_0}{\sqrt{1/a^2(y) - \lambda^2}}\, d\xi,$$

where $r^{*0} = R(\tau')$, $r^* = -R(\tau')$. For large y /27/

$$\tau' = t - \tau_0(x, y) + \lambda\xi - \frac{\lambda}{2x}\eta^2, \quad \frac{x}{\lambda} = S,$$

$$\frac{x}{\lambda} = \int_0^y \frac{dy}{\sqrt{\frac{1}{a^2(y)} - \lambda^2}}, \quad \sqrt{\psi - x^2} = \sqrt{\frac{2x}{\lambda}}\sqrt{t - \tau_0(x, y)}$$

where $ds = \frac{dS}{a(y)}$ is the arc length of the ray, and V_0 is given by /27/

$$V_0 = \sqrt{\frac{S}{a^2(y)\sqrt{\frac{1}{a^2(y)} - \lambda^2}\int_0^y \frac{\lambda dy}{a^2(y)\,(1/a^2(y) - \lambda^2)^{1/2}}}}.$$

It is clear that for large y one can take $V_0(y) = V_0(\infty)$. Since $\frac{\partial}{\partial y} = -\sqrt{\frac{1}{a^2(y)} - \lambda^2}\,\frac{\partial}{\partial \tau'}$ as $y \to \infty$, the relation $P = \frac{F(\tau')}{\sqrt{S}}$ is derived, where

$$F(\tau') = \frac{1}{\pi} V_0(\infty)\sqrt{\frac{1}{a^2(\infty)} - \lambda^2}\,\sqrt{2}\,\frac{\partial}{\partial \tau'}\int_0^{\sqrt{\tau'}} d\eta' \times$$

$$\times \int_{-r_1^*(\tau')}^{r_0^*(\tau')} P_1\, d\xi, \quad P_1 = P_1(|\xi|, \tau').$$

Replacing the linear characteristic τ' by y_1, where $y_1 = \text{const}$ is the equation of the improved one-dimensional characteristics, we have

$$\frac{\partial t}{\partial S} = \frac{1}{a(y)} - \frac{1}{a(y)}\,\alpha^\circ\,\frac{P}{\rho_0 a^2(y)},$$

which, after integration, yields

$$t = \int_0^s \frac{ds}{a(y)} - \alpha^\circ F(y_1)\int_0^s \frac{ds}{\rho_0 a^2(y)\sqrt{s}} + y_1.$$

On the shock front the following equation holds /18/:

$$\alpha^\circ \int_0^y \frac{dy}{\rho_0 a^4(y)\sqrt{\frac{1}{a^2(y)}-\lambda^2}} \int_0^y \frac{dy}{\sqrt{1/a^2(y)-\lambda^2}} = \frac{2\int_0^{y_0} F(\tau')\,d\tau'}{F^2(y)_1},$$

where y_0 is the maximum value of y_1 (§ 6, Chapter II).

For particular functions $\rho_0(y)$ and $a(y)$, the solution along $0y$ in final form can be found by the Smirnov—Sobolev method.

APPENDIX IV

Linear and nonlinear problems of the motion of a conducting and viscous fluid in a half-space

1. Consider the motion of a half-space of ideal compressible fluid due to a shock pressure given on its surface. We take a two-dimensional problem, for which the $0x$ axis lies along the fluid surface and the $0y$ axis points inwards. The boundary condition for the pressure $P(x, y, t)$ on the surface is

$$P(x, 0, t) = \begin{cases} P_1(x, t) & |x| < R(t) \\ 0 & |x| > R(t), \end{cases} \tag{1.1}$$

where $|x| = R(t)$ is the coordinate of the pressure front on the surface.

First take the case $R'(t) = \text{const} = V$, $P_1^0 = \text{const}$ (the problem of an explosion at the surface). In a linear formulation the pressure satisfies

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = \frac{1}{a_0^2}\frac{\partial^2 P}{\partial t^2}, \tag{1.2}$$

where a_0 is the initial velocity of sound in the fluid.

It is always assumed that $R'(t) > a_0$. With a view to generalizing the case to a conducting fluid, the solution of (1.2) (subject to condition (1.1) and zero initial data) is sought using the method of the separation of variables. The solution is taken in the form of the Fourier integral /51/

$$P(x, y, t) = \int_{-\infty}^{\infty} R(y, t, k)\, e^{ikx}\, dk; \tag{1.3}$$

hence, $R(y, t, k)$ satisfies the equation

$$\frac{\partial^2 R}{\partial y^2} - \frac{1}{a_0^2}\frac{\partial^2 R}{\partial t^2} - k^2 R = 0$$

with the boundary condition

$$R|_{y=0} = \frac{1}{2\pi} \int_{-Vt}^{Vt} P_1^0 \, e^{-ikx} dx.$$

The following transform with respect to t is introduced:

$$\overline{X}(y, k, s) = \int_0^\infty R(y, k, t) \, e^{-st} \, dt.$$

Since

$$\int_0^\infty e^{-st} dt \int_{-Vt}^{Vt} P_1^0 \, e^{-ikx} dx = \frac{2P_1^0 V}{s^2 + k^2 V^2},$$

the solution of the equation

$$\frac{d^2\overline{X}}{dy^2} - \left(\frac{s^2}{a_0^2} + k^2\right)\overline{X} = 0$$

for $\overline{X}$ subject to the boundary condition $\overline{X}|_{y=0} = \frac{1}{\pi} \frac{P_1^0 V}{s^2 + k^2 V^2}$ is obtained in the form

$$\overline{X} = C(k, s) \, e^{-y\sqrt{\frac{s^2}{a_0^2} + k^2}},$$

$$C(k, s) = \frac{1}{\pi} \frac{P_1^0 V}{s^2 + k^2 V^2}.$$

The inverse transform is derived from (1.3), so that

$$P(x, y, t) = \int_{-\infty}^{\infty} e^{ikx} dk \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} C(k, s) \; e^{-y\sqrt{\frac{s^2}{a_0^2} + k^2}} \; e^{st} ds \qquad (1.4)$$

and, introducing $\zeta = \frac{s}{a_0 k}$, (1.4) yields

$$P(x, y, t) = \frac{P_1^0 V}{\pi} a_0 \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{k} \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{-|k| y \sqrt{\zeta^2 + 1}} e^{k a_0 \zeta t}}{\zeta^2 a_0^2 + V^2} d\zeta. \qquad (1.5)$$

The solution near a front is found by separating the principal part in the integral with respect to ζ. The solution on the front AB (Figure 8), determined by taking the residue at the points $\zeta = \pm i \frac{V}{a_0}$, is

$$P_M = \frac{P_1^0}{\pi} \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{ikx}}{k} e^{-i|k| y \sqrt{\frac{V^2}{a_0^2} - 1}} \, 2 \sin kVt \, dk$$

or

$$P_M = \frac{P_1^0}{\pi}\int_0^\infty \frac{2\cos kx\, e^{-iky\sqrt{\frac{V^2}{a_0^2}-1}}}{k}\sin kVt\,dk.$$

It can be easily shown that $P_M = P_1^0$. To determine the solution near the segment B_1B_1' of the front (Figure 5), which has the form of the circle

$$a_0 t = \sqrt{x^2 + y^2},$$

we consider the principal part of the integral with respect to ζ by the stationary-phase method. The expression in the exponent is denoted by

$$f(\zeta) = \zeta a_0 t - \frac{|k|}{k} y\sqrt{\zeta^2+1}.$$

The saddle points, the roots of the equation $f'(\zeta)=0$, are $\zeta_{1,2} = \frac{\pm it}{\sqrt{t^2 - y^2/a_0^2}}$. An integration contour λ is chosen for (1.5) in the form of a curve passing through the saddle points at angles $\frac{3\pi}{4}$ and $\frac{\pi}{4}$, obtained from the condition

$$\mathrm{Re} f''(\zeta_{1,2})(\zeta - \zeta_{1,2})^2 = 0.$$

Then it can be shown by the stationary-phase method /51/ or with the aid of Gauss' integral /49/ that

$$\int_\lambda \psi(\zeta) e^{ikf(\zeta)}\,d\zeta = \sum_{1,2}\psi(\zeta_k)\, e^{ikf(\zeta_k)}\sqrt{\frac{2\pi}{k|f''(\zeta_k)|}}\, e^{i\theta_k},$$

$$\theta_1 = \frac{3\pi}{4},\quad \theta_2 = \frac{\pi}{4},\quad \psi(\zeta_{1,2}) = \frac{1}{\zeta_{1,2}^2 a_0^2 + V^2}.$$

When integrating with respect to k, large values of k are important near the front. The integral is therefore taken over k within the limits $-\infty, -N; N, \infty$. The pressure is then given by

$$P(x,y,t) = \frac{P_1^0 V}{\pi} a_0^{-\frac{1}{2}} \int_{-\infty}^{\infty} \frac{1}{2\pi i}\frac{e^{ikx}}{k\sqrt{k}}dk \times$$

$$\times \frac{e^{i\frac{3\pi}{4}} e^{ika_0\sqrt{t^2-\frac{y^2}{a_0^2}}}}{\left(t^2-\frac{y^2}{a_0^2}\right)^{1/4}} \frac{(2\pi)^{\frac{1}{2}} y}{\zeta_1^2 a_0^2 + V^2} + \frac{P_1^0 V}{\pi}\frac{1}{2\pi i} a_0^{-\frac{1}{2}} \times$$

$$\times \int_{-\infty}^{\infty} \frac{e^{ikx}\,dk}{k\sqrt{k}} \frac{e^{i\frac{\pi}{4}} e^{-ika_0\sqrt{t^2-\frac{y^2}{a_0^2}}}}{(\zeta_0^2 a_0^2 + V^2)\left(t^2-\frac{y^2}{a_0^2}\right)^{1/4}} (2\pi)^{\frac{1}{2}} y, \qquad (1.6)$$

where the integration over k is performed for large k (zero is excluded). Near the front $x = a_0\sqrt{t^2 - \frac{y^2}{a_0^2}}$ only the second integral in (1.6) is significant; however, we do not make use of this fact here. Taking into account that $e^{i\frac{3\pi}{4}} = -e^{-i\frac{\pi}{4}}$ and collecting terms in both integrals, the expression

$$P(x, y, t) = \frac{P_1^0 V a_0}{\pi}\frac{1}{2\pi}\left(\frac{2\pi}{\left(t^2 - \frac{y^2}{a_0^2}\right)^{1/2}}\right)^{\frac{1}{2}} y a_0^{-\frac{3}{2}} \frac{1}{\zeta_1^2 a_0^2 + V^2} \times$$

$$\times \int_{-\infty}^{\infty} \frac{e^{ikx}\, 2\sin\left(\frac{\pi}{4} - ka_0\sqrt{t^2 - \frac{y^2}{a_0^2}}\right)}{\sqrt{k}}\, dk$$

is obtained. If the real part is separated and k replaced by $-k$ in the integral from $-\infty$ to $-N$, then

$$\frac{\partial P}{\partial x} = -\frac{P_1^0 V a_0}{\pi}\frac{1}{2\pi}\left(\frac{2\pi}{\left(t^2 - \frac{y^2}{a_0^2}\right)^{1/2}}\right)^{1/2} y a_0^{-\frac{3}{2}} \frac{1}{\zeta_1^2 a_0^2 + V^2} \times$$

$$\times \int_0^{\infty} \frac{2\sin\left(\frac{\pi}{4} + ka_0\sqrt{t^2 - \frac{y^2}{a_0^2}} - kx\right)}{\sqrt{k}}\, dk,$$

where the integration over k is also performed from 0 to ∞.

Expanding the sine term and making use of the fact that

$$\int_0^{\infty} \frac{\sin kt'}{\sqrt{k}}\, dk \quad \text{and} \quad \int_0^{\infty} \frac{\cos kt'}{\sqrt{k}}\, dk \text{ equal } \sqrt{\frac{\pi}{2t'}},$$

we obtain

$$\frac{\partial P}{\partial x} = -\frac{P_1^0 V a_0}{\pi}\frac{1}{2\pi}\left(\frac{2\pi}{\left(t^2 - \frac{y^2}{a_0^2}\right)^{\frac{3}{2}}}\right)^{\frac{1}{2}} y a_0^{-\frac{3}{2}} \frac{t^2 - \frac{y^2}{a_0^2}}{a_0^2 t^2 - x^2\frac{V^2}{a_0^2}}\sqrt{\pi} \times$$

$$\times 2\sqrt{\frac{1}{a_0\sqrt{t^2 - \frac{y^2}{a_0^2}} - x}}.$$

Since $t^2 - \frac{y^2}{a_0^2} = \frac{x^2}{a_0^2}$, the expression for $\frac{\partial P}{\partial x}$ agrees with the value on BB' given by (3.1) of Chapter I.

The case of arbitrary $R(t)$ and P_1 is now considered. The solution is sought on the wave front AB (Figure 8), which represents the envelope of elementary waves arising on the surface. The equation of these waves is obtained as the envelope of plane waves

$$t - f(x') - \xi(x - x') - y\sqrt{\xi^2 - \frac{1}{a_0^2}} = 0 \qquad (1.7)$$

in terms of x' and ξ. Here $t = f(x')$ is the inverse function of $x' = R(t)$, and x' represents the point C of the surface (lying on the ray CM (Figure 8)) through which the front passed at time $t = f(\xi')$. The cosine of the angle between the ray CM and $0x$ is ξ.

The envelope of (1.7) gives, in terms of x' and ξ, the equation of AB in the form

$$\begin{gathered} t - f(x') - \xi(x - x') - y\sqrt{\xi^2 - \frac{1}{a_0^2}} = 0, \\ -f'(x') + \xi = 0, \\ x - x' + y\frac{\xi}{\sqrt{\xi^2 - 1/a_0^2}} = 0. \end{gathered} \tag{1.8}$$

The solution of (1.2) is now sought subject to the boundary condition (1.1) by the method of the separation of variables; hence

$$P = \int_{-\infty}^{\infty} e^{-ikx}\, dk\, \frac{1}{2\pi i} \int_{s-i\infty}^{s+i\infty} \bar{X}\, e^{st}\, ds$$

$$\bar{X} = C(k, s)\, e^{-y\sqrt{\frac{s^2}{a_0^2} + k^2}}, \tag{1.9}$$

where

$$C(k, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx'}\, dx' \int_{0}^{\infty} e^{-st'} P_1(x', t')\, dt'. \tag{1.10}$$

When calculating the integrals near a wave front on AB by the stationary-phase method the only important data are the values of x' and t' which are given by (1.8), where $x' = \eta_s$, $t' = f(\eta_s)$. The principal part of P_1 can therefore be removed from the integral. The integral over t' then gives

$$\int_{f(x')}^{\infty} e^{-st'} P_1\, dt' = P_1(x', f(x'))\, \frac{e^{-sf(x')}}{s},$$

and so the pressure on AB is

$$P = \frac{1}{2\pi i} P_1 \int_{\sigma-i\infty}^{\sigma+i\infty} ds\, \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx}\, dk\, e^{st}\, e^{-y\sqrt{\frac{s^2}{a_0^2} + k^2}} \int_{-\infty}^{\infty} \frac{e^{ikx}\, e^{-sf(x')}}{s}\, dx',$$

where for $x > 0$ on the wave front the values $x' > 0$ are significant.

The Laplace transform $\bar{P}(x, y, s)$ of P is

$$\bar{P} = \frac{P_1(x_s', f(x_s'))}{s} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{0}^{\infty} e^{-i\omega f(x') - ik(x - x') - y\sqrt{k^2 - \frac{\omega^2}{a_0^2}}}\, dx', \tag{1.11}$$

where $s = i\omega$. Suppose $x' = \eta$, $\frac{k}{\omega} = \xi$, $\xi = \frac{1}{a_0}\sin\theta$, where θ is the angle between the ray CM and $0y$ /52/.

Denoting the expression in the exponent of (1.11) by $-i\omega T$ and applying the stationary-phase method we have $T_\xi = 0$, $T_\eta = 0$ or

$$\xi_s = f'(\eta_s),$$
$$\eta_s = x - y \operatorname{tg} \theta_s, \tag{1.12}$$

where (1.12) agrees with (1.8), and the expression in the exponent equals $-i\omega t$ on the wave front (1.8). Introducing new variables $\xi - \xi_s = x_0$, $\eta - \eta_s = y_0$, we obtain from (1.11)

$$\overline{P} = \frac{1}{2\pi} \frac{P_1}{i\omega} \omega e^{-i\omega T_s} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\frac{\omega}{2}(T_{\xi\xi_s} x_0^2 + 2T_{\xi\eta_s} x_0 y_0 + T_{\eta\eta_s} y_0^2)} dx_0 dy_0.$$

The integral is evaluated in /52/ to yield

$$\overline{P} = (T_{\xi\eta_s}^2 - T_{\xi\xi_s} T_{\eta\eta_s})^{-\frac{1}{2}} \frac{P_1}{i\omega} e^{-i\omega T_s}. \tag{1.13}$$

If the inversion formula is used to transform to the pressure P, then

$$P = \frac{1}{\sqrt{1 + \frac{y}{a_0} f''(\eta_s) \frac{1}{\cos^3 \theta_s}}} P_1(\eta_s, f(\eta_s)) \sigma(t - f(\eta_s)), \tag{1.14}$$

where $\sigma(t)$ is Heaviside's unit function. Expression (1.14) agrees with (5.25) of Chapter I /27/, where $f(\eta_s) = t^*$, $\sin \theta = \frac{a_0}{R'(t^*)}$.

Thus, by separating the variables, solutions which were found in /27/ by another method were obtained on the wave fronts. In the axisymmetric case /52/, the right-hand side of (1.14) should be multiplied by $\sqrt{\frac{\eta_s}{x}}$.

2. Consider the motion of a half-space of infinitely conducting compressible fluid in a magnetic field. Initially, the fluid is at rest and the magnetic field is uniform and parallel to the $0x$ axis so that $B_y = 0$, $B_x = B^\circ$. A shock wave forms as a result of an explosion in the air, which is assumed to be nonconducting. The shock is reflected from the fluid, and since the fluid impedance $\rho_0 a_0$ is much larger than that of air, the pressure at the surface may be considered to be the same as in the reflection of a shock wave from a rigid nonconducting wall. Thus, the pressure in the air at the interface is known from (1.1).

Suppose b_x, b_y denotes the disturbed value of the magnetic field intensity. At the interface $y=0$ we have boundary conditions /53/ expressing continuity of the total pressure and of the normal component of the magnetic field intensity (/11/, Vol. 6):

$$P + \frac{B^\circ b_x}{4\pi} = P_1 + \frac{B^\circ b_{x_1}}{4\pi},$$
$$b_{y_1} = b_y \tag{2.1}$$

(the subscript 1 refers to the air). In the air Maxwell's equations and the equations of gas dynamics are solved separately, and since the conductivity $\gamma=0$, the current density $j=0$ and rot $\overline{B}=0$, div $\overline{B}=0$.

The magnetic susceptibility $\mu=1$ in both media. Hence in the upper half-space

$$\overline{B} = \operatorname{grad}\varphi,\ \Delta\varphi=0,\ \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

The linearized equations of motion of a conducting fluid with infinite conductivity have the form /49/

$$\frac{\partial b_x}{\partial t} = -B^\circ\frac{\partial v_y}{\partial y},\ \frac{\partial v_x}{\partial t} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x},\ \frac{\partial v_y}{\partial t} = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}+\frac{B^\circ}{4\pi\rho_0}\left(\frac{\partial b_y}{\partial x}-\frac{\partial b_x}{\partial y}\right),$$

$$\frac{\partial b_y}{\partial t} = B^\circ\frac{\partial v_x}{\partial y},\ \frac{\partial P}{\partial t} = -\rho_0 a_0^2\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right). \tag{2.2}$$

where v_x and v_y are the components of the velocity vector.

If we introduce the symbolic notation $\frac{\partial}{\partial x}=\xi_1$, $\frac{\partial}{\partial y}=\xi_2$, $\frac{\partial}{\partial t}=\tau$, all variables, except the pressure P, can be eliminated from system (2.2) to yield

$$b_x = -\frac{B^\circ\xi_2}{\tau}v_y,\quad b_y=\frac{B^\circ\xi_2}{\tau}v_x,\ v_x=-\frac{1}{\rho_0}\frac{\xi_1}{\tau}P,$$

$$v_y = \frac{-\frac{1}{\rho_0}\xi_2}{\tau-\frac{1}{4\pi\rho_0}B^{\circ 2}\left(\frac{\xi_1^2}{\tau}+\frac{\xi_2^2}{\tau}\right)}P, \tag{2.3}$$

where the pressure P obeys the fourth-degree equation

$$((\tau^2-a_1^2(\xi_1^2+\xi_2^2))(\tau^2-a_0^2\xi_1^2)-a_0^2\xi_2^2\tau^2)P=0, \tag{2.4}$$

$a_1^2=\frac{B^{\circ 2}}{4\pi\rho_0}$ being the square of the Alfvén velocity. If the solution of (2.4) is sought in the form of the plane waves

$$P = e^{-i\omega t+ikx+i\beta y}, \tag{2.5}$$

we obtain the equation of the surface G, defined by the normals to the wave (2.5), in the form

$$G=0,\ (-\omega^2+a_1^2(k^2+\beta^2))(-\omega^2+a_0^2k^2)-a_0^2\beta^2\omega^2=G \tag{2.6}$$

or, setting $-i\omega=s$,

$$\beta^2 = -\frac{(s^2+a_1^2k^2)(s^2+a_0^2k^2)}{a_0^2s^2+a_1^2s^2+a_1^2a_0^2k^2} \tag{2.7}$$

The equation of a ray surface or of a constant phase surface is found from the condition that the rays are perpendicular to the above-defined surface /49/, namely

$$x = -\frac{\dfrac{\partial G}{\partial k}}{\dfrac{\partial G}{\partial \omega}} t, \qquad y = -\frac{\dfrac{\partial G}{\partial \beta}}{\dfrac{\partial G}{\partial \omega}} t \tag{2.8}$$

or

$$x = \frac{-\omega^2 (a_0^2 + a_1^2) k + a_0^2 a_1^2 k (k^2 + \beta^2) + a_0^4 a_1^2 k^3}{2\omega^3 - \omega (a_0^2 + a_1^2)(k^2 + \beta^2)} t,$$

$$y = -\frac{a_1^2 \beta (-\omega^2 + a_0^2 k^2) - a_0^2 \beta \omega^2}{2\omega^3 - \omega (a_0^2 + a_1^2)(k^2 + \beta^2)} t. \tag{2.9}$$

In virtue of the homogeneity of G we have

$$k \frac{\partial G}{\partial k} + \beta \frac{\partial G}{\partial \beta} + \omega \frac{\partial G}{\partial \omega} = 0. \tag{2.10}$$

Equation (2.9) defines an elementary wave from a point source, introduced by Friedrichs /54/. When $a_0 > a_1$, this wave has the form of an oval (Figure 19), which is an accelerated gas-dynamic wave, and two triangular-shaped ones, corresponding to decelerated waves. The characteristic points of Figure 19 are given by

$$y = 0, \quad x = a_0 t, \quad k = \frac{\omega}{a_0}, \quad \beta = 0; \quad y = 0, \quad x = a_1 t, \quad k = \frac{\omega}{a_1}, \quad \beta = 0;$$

$$y = 0, \quad x = \frac{t}{\sqrt{a_0^{-2} + a_1^{-2}}}, \quad k = \omega \sqrt{a_0^{-2} + a_1^{-2}}, \quad \beta = \infty;$$

$$x = 0, \quad y = \sqrt{a_0^2 + a_1^2}\, t, \quad k = 0, \quad \beta = \frac{\omega}{\sqrt{a_0^2 + a_1^2}}.$$

(See "Methods of Mathematical Physics" by R. Courant [Interscience, N.Y. 1953]).

The variables $\beta_1 = \dfrac{\beta}{k}$, $\omega_1 = \dfrac{\omega}{k}$ are introduced and (2.9) is written as

$$x = \omega_1 t - \frac{\beta_1}{\dfrac{\partial \beta_1}{\partial \omega_1}} t,$$

$$y = \frac{t}{\dfrac{\partial \beta_1}{\partial \omega_1}}. \tag{2.11}$$

Since the front BB' is influenced only by the initial values of P_1 and $R(t)$, the solution near BB' (Figure 5) is identical to that of constant P_1 and

$R' = V$. The solution of (2.4) is taken in the form

$$P = \int_{-\infty}^{\infty} e^{ikx}\, dk\, \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} A(k, s)\, e^{i\beta y} e^{st}\, ds, \tag{2.12}$$

β being found from (2.7), where $\frac{\beta}{k} = \beta_1$, $\frac{s}{k} = \zeta a_0$,

$$i\beta_1 = -\sqrt{\frac{\left(\zeta^2 + \frac{a_1^2}{a_0^2}\right)(\zeta^2 + 1)}{\zeta^2 + \frac{a_1^2}{a_0^2}\zeta^2 + \frac{a_1^2}{a_0^2}}}. \tag{2.13}$$

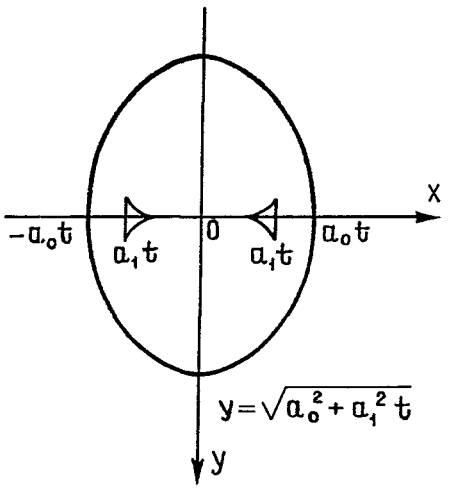


FIGURE 19.

Similarly, the following notation is introduced in the upper half-space:

$$\varphi = \int_{-\infty}^{\infty} e^{ikx}\, dk\, \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \varphi_1 e^{y|k|}\, e^{st}\, ds, \tag{2.14}$$

where the exponent containing y is chosen to satisfy

$$\Delta\varphi = 0,\ k^2 + \beta^2 = 0.$$

In our case for the half-space $y > 0$ only the accelerated wave (2.9) is present (/11/, Vol. 6).

The components of the magnetic field intensity are

$$b_{1x}=\frac{\partial\varphi}{\partial x},\quad b_{1y}=\frac{\partial\varphi}{\partial y},\quad \bar{b}_{1x}=ik\varphi_1,\quad \bar{b}_{1y}=|k|\varphi_1. \tag{2.15}$$

Substituting these expressions in (2.1) and using (2.3), we have

$$\bar{b}_x=\frac{-\frac{B^\circ}{\rho_0}\beta^2\bar{P}}{s^2+a_1^2(k^2+\beta^2)},$$

$$\bar{b}_y=\frac{\frac{B^\circ}{\rho_0}k\beta\,\bar{P}}{s^2+a_1^2(k^2+\beta^2)},$$

$$|k|\varphi_1=\bar{b}_y,\quad \bar{P}+\frac{B^\circ}{4\pi}\bar{b}_x=\bar{P}_1+\frac{B^\circ}{4\pi}ik\varphi_1, \tag{2.16}$$

where the bar denotes the double transform (2.12) in which $\bar{P}=A$, and in (2.12) P should be replaced by any of the functions b_x, b_y, and $\bar{P}$ by $\bar{b}_x$, $\bar{b}_y$. From (2.16) $\bar{P}$ is derived in the form

$$\bar{P}=\frac{1}{1-\frac{a_1^2\beta^2+a_1^2\,i\,|k|\,\beta}{s^2+a_1^2(k^2+\beta^2)}}\bar{P}_1=Z(k,s)\bar{P}_1, \tag{2.17}$$

where $\bar{P}_1$ for constant P_1 and V is given by the inverse transform of (2.12) and agrees with the corresponding expression of §1, namely

$$\bar{P}_1=\frac{1}{\pi}\frac{P_1^\circ V}{s^2+k^2V^2}. \tag{2.18}$$

The solution obtained in the half-space $y>0$ vanishes outside the wave fronts, whereas for $y<0$ it is nonzero everywhere and should therefore cause motion at $y>0$ outside the wave fronts. This situation is eliminated by introducing a finite conductivity $\sigma=\frac{1}{\eta}$ in the half-space $y>0$. Equations (2.2) remain valid with the exception of the first two, which become /54/

$$\frac{\partial b_x}{\partial t}=-B^\circ\frac{\partial v_y}{\partial y}-\frac{\eta}{4\pi}\left(\frac{\partial^2 b_y}{\partial x\partial y}-\frac{\partial^2 b_x}{\partial y^2}\right),$$

$$\frac{\partial b_y}{\partial t}=B^\circ\frac{\partial v_y}{\partial x}+\frac{\eta}{4\pi}\left(\frac{\partial^2 b_y}{\partial x^2}-\frac{\partial^2 b_x}{\partial x\partial y}\right).$$

Eliminating b_x, b_y, v_x, v_y from the system, the following equation is derived for the pressure:

$$\Big((\tau^2-a_0^2\xi_1^2)(\tau^2-a_1^2(\xi_1^2+\xi_2^2))-a_0^2\xi_2^2\tau^2+$$

$$+\tau(\xi_1^2+\xi_2^2)\frac{\eta}{4\pi}(a_0^2\xi_2^2-\tau^2+a_0^2\xi_1^2)\Big)P=0.$$

If a solution is taken in the form

$$P = e^{-i\omega t + ikx + i\beta y},$$

a biquadratic equation results for β, whose roots for small η are

$$\beta_1 = k\sqrt{4\pi\frac{a_1^2\omega_1^2 + a_0^2\omega_1^2 - a_0^2a_1^2}{-a_0^2 i\omega_1\eta}},$$

$$\beta_2^2 = -k^2\frac{(\omega_1^2 - a_1^2)(\omega_1^2 - a_0^2)}{a_1^2\omega_1^2 + a_0^2\omega_1^2 - a_0^2a_1^2},$$

where β_2 corresponds to β in (2.7), and $\omega_1 = \frac{\omega}{k}$.

Thus, a finite conductivity gives rise to an additional term $e^{i\beta_1 y}$ in the solution. Consider a solution in the form

$$\bar{P} = Ae^{i\beta_2 y} + Be^{i\beta_1 y},$$

where

$$\bar{b}_x = -\frac{B^\circ\beta^2\frac{1}{\rho}\bar{P}}{-\omega^2 + a_1^2(k^2+\beta^2) - \frac{i\omega\eta}{4\pi}(k^2+\beta^2)},$$

$$\bar{b}_y = \frac{B^\circ k\beta\frac{1}{\rho}\bar{P}}{-\omega^2 + a_1^2(k^2+\beta^2) - \frac{i\omega\eta}{4\pi}(k^2+\beta^2)},$$

and $\bar{P}$, $\bar{b}_x$, $\bar{b}_y$ are double Fourier transforms with respect to t and x of the form (2.12) for $\beta = \beta_{1-2}$ respectively.

If b_{1x} and b_{1y} are given by (2.14) and (2.15) in the upper half-space $y < 0$ then boundary conditions (2.1) give

$$\frac{B^\circ k\beta_2\frac{1}{\rho}A}{-\omega^2 + a_1^2(k^2+\beta_2^2) - \frac{i\eta\omega}{4\pi}(k^2+\beta_2^2)} + \frac{B^\circ k\beta_1\frac{1}{\rho}B}{-\omega^2 + a_1^2(k^2+\beta_1^2) - \frac{i\eta\omega}{4\pi}(k^2+\beta_1^2)} = -k\varphi,$$

$$\frac{-\frac{B^{\circ 2}}{4\pi}\beta_2^2\frac{1}{\rho}A}{-\omega^2 + a_1^2(k^2+\beta_2^2) - \frac{i\eta\omega}{4\pi}(k^2+\beta_2^2)} - \frac{\frac{B^{\circ 2}}{4\pi}\beta_1^2\frac{1}{\rho}B}{-\omega^2 + a_1^2(k^2+\beta_1^2) - \frac{i\eta\omega}{4\pi}(k^2+\beta_1^2)} + A + B = \frac{B^\circ}{4\pi}ik\varphi + \bar{P}_1.$$

As $\eta \to 0$, $\beta_1 \to i\infty$. Since the conductivity is finite, we require that P be continuous at $y = 0$ i.e.,

$$A + B = \bar{P}_1.$$

Now, as $\eta \to 0$ the term containing B in the first equation of the above system drops out, and in the second equation terms with B cancel in the limit $\beta_1 \to \infty$, and A reduces to its previous value in (2.17). Thus, whenever $e^{i\beta_1 y}$ is small, i. e., in the whole half-space $y > 0$ with the exception of a narrow region near the boundary $y \sim \sqrt{\eta}$, the solution has the same form as for infinite σ; near the boundary it contains a term with $e^{i\beta_1 y}$ and so will differ somewhat. This is a typical example of a boundary layer, which is well known in the theory of viscous fluids.

The solution on BB' (Figure 5) is determined by investigating (2.12) by the stationary-phase method. With the notation $f(\zeta) = \beta_1 y - i\zeta a_0 t$, the calculations of § 1 may be repeated to yield the stationary points

$$f'(\zeta_{1,2}) = 0, \quad \text{where} \quad \zeta = -\frac{i\omega_1}{a_0}$$

or

$$\frac{\partial \beta_1}{\partial \omega_1} y - t = 0, \tag{2.19}$$

i. e., a condition agreeing with the equation (2.11) of BB'. In addition to the equation of the surface defined by the normals (2.13) in the form

$$\beta_1 = \frac{1}{a_0} \sqrt{\frac{(\omega_1^2 - a_1^2)(\omega_1^2 - a_0^2)}{\omega_1^2 + \frac{a_1^2}{a_0^2}\omega_1^2 - a_1^2}}, \tag{2.20}$$

equation (2.19) gives a condition for determining ω_{1s} at the stationary points. It can be easily seen that $\frac{\partial^2 \beta_1}{\partial \omega_1^2}$ is real for $\omega = \omega_{1s}$. The contour angles at these points are denoted by θ_1, θ_2. Then, by the stationary-phase method, the inner integral of (2.12) is

$$-\frac{1}{2\pi i} k \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{1}{\pi} \frac{P_1^0 V e^{i(kx + \beta y - \omega t)}}{-k^2\omega^2 + k^2 V^2} i d\omega_1 =$$

$$= \frac{a_0}{i} \frac{1}{2\pi i} \frac{1}{-\omega_{1s}^2 + V^2} e^{ikf(\omega_{1s})} \sqrt{\frac{2\pi}{k|f''(\omega_{1s})|}} e^{i\theta_1} +$$

$$+ \frac{a_0}{i} \frac{1}{2\pi i} \frac{1}{-\omega_{2s}^2 + V^2} e^{ikf(\omega_{2s})} \sqrt{\frac{2\pi}{k|f''(\omega_{2s})|}} e^{i\theta_2},$$

where, according to (2.2), $\omega_{1s} > 0$, $\omega_{2s} = -\omega_{1s}$. The angles $\theta_{1,2}$ are

$$\theta_{1,2} = \frac{\pi}{4} - \frac{1}{2} \arg f''(\omega_{1,2s}), \quad \omega_{1s} > 0, \; f(\omega_{1s}) < 0, \; f(\omega_{2s}) > 0.$$

From (2.12) we obtain, similarly to (1.6),

$$P = \frac{P_1^0 V'}{\pi} \frac{1}{2\pi} \sqrt{\frac{2\pi}{|f''(\omega_{1s})|}} \int_{-\infty}^{\infty} Z \frac{e^{ik(x+f)}}{-\omega_{1s}^2 + V'^2} \frac{e^{i\frac{\pi}{4}}}{k \sqrt{k}} dk, \quad Z = re^{i\varphi}, \tag{2.21}$$

where the term in θ_2 is omitted, since it is insignificant near BB' for $x = -f$. In addition (§ 1)

$$\frac{\partial P}{\partial x} = \frac{P_1^0 V}{\pi} r \cos\varphi \frac{i}{2\pi} \sqrt{\frac{2\pi}{|f''(\omega_{1s})|}} \frac{1}{-\omega_{1s}^2 + V'^2} 2a_0 (\sin\theta_1 + \cos\theta_1) \sqrt{\frac{\pi}{2(-f-x)}},$$

ω_{1s} being the velocity of the plane wave (2.5) along $0x$. Integrating, we find that

$$P = -\frac{P_1^0 V}{\pi} Z(\omega_{1s})\, a_0^2 \sqrt{\frac{1}{|f''(\omega_{1s})|}}\, 2 \frac{1}{-\omega_{1s}^2 + V'^2} (\sin\theta_1 + \cos\theta_1) \sqrt{-f-x} \tag{2.22}$$

near BB', where $\theta_1 = \frac{\pi}{4}$, $\omega_{1s} \gg V$. $r\cos\varphi$ is denoted by Z.

Consider now the pressure on the front AB, which represents the envelope (2.11) of the waves at time t' (where x is replaced by $x - R(t')$ and t by t').

Solution (2.12) is written in a form similar to (1.9):

$$P = \int_{-\infty}^{\infty} e^{ikx} dk \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{i\beta y} e^{st} A ds, \tag{2.23}$$

where $A = Z\overline{P}_1$. When using the stationary-phase method, the smoothly varying part can be taken outside the integral. The expression for $\overline{P}$ therefore takes the form

$$\overline{P} = P_1 \int_{-\infty}^{\infty} dk \frac{1}{2\pi} \int_{-\infty}^{\infty} Z \frac{e^{ik(x-x') + i\beta y - sf(x')}}{s} dx', \tag{2.24}$$

$$P_1 = P_1(x', f(x')).$$

With the notation

$$x' = \eta, \quad \frac{k}{\omega} = \xi, \quad s = -i\omega,$$

$$T = \xi(x - \eta) + \frac{\beta}{\omega} y + f(\eta),$$

the stationary-phase method gives

$$T_\xi = 0, \quad T_\eta = 0 \text{ or}$$

$$f'(\eta_s) = \xi_s,$$

$$x - \eta_s + \frac{\partial \beta'}{\partial \xi} y = 0,$$

$$\beta_1 = \frac{\beta}{\omega}. \tag{2.25}$$

Hence, from (2.25) and the equation $T = t$ of the plane waves the envelope of the plane waves is derived in terms of ξ and η, i.e., the line AB (Figure 5). In (2.24) $Z = Z\left(\frac{|k|}{k}, \xi\right)$, and therefore for calculations in which $k>0$ and $k<0$ one is left with $Re\, Z = r\cos\varphi$, henceforth denoted by Z.

If the variables $x_0 = \xi - \xi_s$, $y_0 = \eta - \eta_s$ are introduced in (2.24), we obtain

$$\overline{P} = \frac{P_1}{2\pi} Z \frac{1}{i\omega} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} e^{-i\omega T}\, d\eta$$

or

$$\overline{P} = \frac{P_1}{2\pi} Z \frac{1}{i} e^{-i\omega T_s} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\frac{\omega}{2}\left(T_{\xi\xi}x_0^2 + 2T_{\xi\eta}x_0y_0 + T_{\eta\eta}y_0^2\right)} dx_0 dy_0, \tag{2.26}$$

where T_s is the value of T at the point $\xi = \xi_s$, $\eta = \eta_s$. This integral has the value /52/

$$\overline{P} = \frac{P_1 Z(\xi_s, \eta_s)}{i\omega} \left(T_{\xi\eta}^2 - T_{\xi\xi}T_{\eta\eta}\right)^{-\frac{1}{2}} e^{-i\omega T_s}. \tag{2.27}$$

The inverse transform yields

$$P = P_1 Z(\xi_s, \eta_s) \frac{1}{\sqrt{T_{\xi\eta}^2 - T_{\xi\xi}T_{\eta\eta}}}\, \sigma(t - T_s). \tag{2.28}$$

The derivatives of T are

$$T_\xi = x - \eta + \frac{\partial\beta_1}{\partial\xi} y,\quad T_{\xi\xi} = \frac{\partial^2\beta_1}{\partial\xi^2} y,\quad T_{\xi\eta} = -1,\quad T_{\eta\eta} = f''(\eta),$$

so that

$$P = P_1 Z(\xi_s, \eta_s) \frac{1}{\sqrt{1 - \frac{\partial^2\beta_1}{\partial\xi^2}\, y f''(\eta_s)}}. \tag{2.29}$$

The equation of the front AB or the equations of the rays have the form (from (2.25))

$$\begin{gathered} -t + f(\eta) + \xi(x - \eta) + \beta_1 y = 0, \\ \xi = f'(\eta), \\ x - \eta + \frac{\partial\beta_1}{\partial\xi} y = 0. \end{gathered} \tag{2.30}$$

Along each ray $\theta = \text{const}$, $\operatorname{ctg}\theta = \frac{y}{x - \eta}$ we have $\operatorname{ctg}\theta = -\frac{1}{\partial\beta_1(\xi)/\partial\xi}$ and $\xi = \text{const}$ (the inclination of the normal to the front is constant along a ray).

The radius of curvature of the wave front is now determined. It is clear that $R = \frac{1}{\frac{\partial\alpha}{\partial s}}$ (where α is the angle between the tangent to the wave front

and $0x$, and s is the arc length of the front AB) and

$$\frac{\partial \alpha}{\partial s} = -\frac{\partial \alpha}{\partial x}\cos\alpha + \frac{\partial \alpha}{\partial y}\sin\alpha \tag{2.31}$$

or

$$\frac{\partial \alpha}{\partial s} = -\frac{\partial \alpha}{\partial x}\frac{\beta_1}{\sqrt{\xi^2+\beta_1^2}} + \frac{\partial \alpha}{\partial y}\frac{\xi}{\sqrt{\xi^2+\beta_1^2}}. \tag{2.32}$$

Since

$$\sin\alpha = \frac{\xi}{\sqrt{\xi^2+\beta_1^2}},\quad \sin\alpha = a_2 f'(\eta), \tag{2.33}$$

where $a_2 = 1/\sqrt{\xi^2+\beta_1^2}$ is the wave front (phase) velocity,

$$\frac{\partial \alpha}{\partial s} = -\frac{\frac{\partial \xi}{\partial x}\beta_1^2 - \xi\beta_1\frac{\partial \beta_1}{\partial x} - \operatorname{tg}\alpha\left(\frac{\partial \xi}{\partial y}\beta_1^2 - \xi\beta_1\frac{\partial \beta_1}{\partial y}\right)}{(\xi^2+\beta_1^2)^{1/2}}. \tag{2.34}$$

From (2.30)

$$\frac{x-\eta}{y} = -\frac{\partial \beta_1}{\partial \xi},\ \xi = f'(\eta); \tag{2.35}$$

consequently

$$\frac{\partial \eta}{\partial x} = \frac{1}{1 - y\frac{\partial^2 \beta_1}{\partial \xi^2}f''(\eta)}. \tag{2.36}$$

After calculating the derivatives in the formula for $\frac{\partial \alpha}{\partial s}$, we obtain

$$\frac{\partial \alpha}{\partial s} = -\frac{\sin^2\theta - 2\cos^2\theta\sin\theta\frac{\partial \beta_1}{\partial \xi} + \cos^2\theta\left(\frac{\partial \beta_1}{\partial \xi}\right)^2}{1 - y\frac{\partial^2 \beta_1}{\partial \xi^2}f''(\eta)}. \tag{2.37}$$

The numerator of this expression is replaced by $\theta = \text{const}$ along a ray. Hence, the ratio of the radius of curvature at the point M of the front to that at the point C of the surface (Figure 5) is

$$\frac{R}{R_1} = 1 - y\frac{\partial^2 \beta_1}{\partial \xi^2}f''(\eta). \tag{2.38}$$

Formula (2.30) now becomes

$$\frac{P}{P_1 Z(\xi_s, \eta_s)} = \sqrt{\frac{R_1}{R}}, \tag{2.39}$$

i.e., the known law for the intensity of sound.

In the axisymmetric case with a radial field $B_x = B^\circ$, $B_y = 0$, one should add $-B^\circ \frac{v_y}{x}$ to the right-hand side of the second equation in (2.2), and $\frac{v_x}{x}$ to the right-hand side of the last equation. The solution is complicated by the fact that when the variables are separated the equation for x is of fourth order. The solution in the general axisymmetric problem is found on the wave front AB by a ray representation used by Babich for an anisotropic elastic medium (Voprosy dinamicheskoi teorii (Problems in Dynamics), Vol. 5, Leningrad. 1961).

The equation (2.30) of the rays for the axisymmetric case is written with respect to x_1 and x_2 axes in the plane of the fluid surface and an $0x_3$ axis inwards, i. e.,

$$-t + f(\eta) + \xi(r - \eta) + \beta_1 y = 0, \tag{2.40}$$

$$\xi = f'(\eta), \quad r - \eta + \frac{\partial \beta_1}{\partial \xi} y = 0, \tag{2.41}$$

where $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = y$. The solution near the front (2.41) ($t = \tau$) is sought in the form

$$P = P^\circ \sigma(t - \tau).$$

With the ray representation, the pressure on the front AB is

$$P^\circ = \frac{f(\eta)}{\sqrt{J}}, \tag{2.42}$$

where $\eta = \text{const}$ is the ray equation (2.41), and

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(t, \eta, \varphi)}.$$

The Jacobian is given by

$$J = r\left(1 - y f''(\eta) \frac{\partial^2 \beta_1}{\partial \xi^2}\right)$$

and the pressure on the front AB at the point M is

$$P^\circ = P_1 Z(\xi_s, \eta_s) \frac{1}{\sqrt{1 - y f''(\eta) \frac{\partial^2 \beta_1}{\partial \xi^2}}} \sqrt{\frac{\eta_s}{r}},$$

where $P_1 Z(\xi_s, \eta_s)$ is the value of P° at the point C on the surface. Thus, as for an isotropic fluid, the pressure is inversely proportional to the square root of the radius of curvature of the front. The same result is obtained by considering the energy conservation of a ray tube.

The nonlinear problem on the shock wave BB' (Figure 5) is solved by replacing the linear characteristics by improved ones in solution (2.22). Near the front BB' (2.11) takes the form

$$x = \left(\omega_2 - \frac{\beta_1}{\frac{\partial \beta_1}{\partial \omega_2}}\right)(t-\tau),$$

$$y = \frac{t-\tau}{\frac{\partial \beta_1}{\partial \omega_2}}, \tag{2.43}$$

which defines a transformation from x, y into τ, ω_2 for a given t, where $\tau=0$ is the front BB' in the linear problem and $\frac{\omega}{k} = \omega_2$. The equation of the characteristic surface in these variables has the form /54/

$$\frac{1-\frac{\partial \tau}{\partial t}}{\sqrt{\frac{1+\beta_1^2}{\omega_2^2} + 2\frac{\partial \tau}{\partial \omega_2}\left(\frac{\omega^{-2}\beta_1'}{\beta_1' t} - \frac{\omega_2 \beta_1'^2 - \beta_1 \beta_1'}{t\beta_1 \beta_1'}\frac{\beta_1^2}{\omega_2^2}\right)}} = c + u_n, \tag{2.44}$$

where $u_n = \frac{P}{Bn} c_0$ is the velocity normal to the characteristic, $\tau(t, \omega_2)$ is the equation of the surface, τ being small near the front. The velocity c of the surface obeys the equation /54/

$$\rho c^4 - (\rho a^2 + \mu_e H^2) c^2 + a^2 \mu_e H_n^2 = 0,$$

where

$$H_x = B^\circ + b_x, H_y = b_y, H_n = (B^\circ + b_x)\cos\theta - b_y \sin\theta, \theta = \theta_0 + \theta';$$

θ_0 is the angle between the normal and the characteristic (2.11), and from (2.43)

$$\operatorname{ctg}\theta = -\frac{\partial y}{\partial x}; \operatorname{ctg}\theta_0 = \frac{1}{\beta_1}; \theta' = -\frac{\sin^2\theta_0}{\operatorname{tg}\theta_0}\frac{\partial \tau}{\partial \omega_2}\frac{\omega^2 \beta_1'^2}{\beta_1 \beta_1' t}.$$

Retaining first-order small quantities we have $c = c_0 + c_1$, where

$$\rho_0 c_0^4 - (\rho_0 a_0^2 + \mu_e B^{-2}) c_0^2 + a_0^2 \mu_e B^{-2} \cos\theta_0 = 0,$$

$$c_1 = \frac{((n-1) a_0^2 c_0^2 - (2-n) a_0^2 a_1^2 \cos^2\theta_0 - c_0^2 a_1^2) +}{2 c_0 (2c_0^2 - a_0^2 - a_1^2)}$$

$$2a_1^2 \frac{\frac{2a_1^2 c_0^2 \sin^2\theta_0 - a_1^2 a_0^2 \sin^2\theta_0 \cos^2\theta_0 + c_0^2 a_1^2 \sin^2\theta_0}{a_0^2 (c_0^2 - a_1^2 \cos^2\theta_0)}}{2c_0 (2c_0^2 - a_0^2 - a_1^2)} \frac{P}{Bn} +$$

$$+ \frac{2a_1^2 a_0^2 \sin\theta_0 \cos\theta_0}{2c_0 (2c_0^2 - a_0^2 - a_1^2)} \theta'. \tag{2.45}$$

In equation (2.44) terms on the left-hand side containing the derivatives $\frac{\partial \tau}{\partial \omega_2}$ cancel with the term on the right-hand side containing θ'. The equation (2.44) of the characteristics is then one-dimensional in τ and t

(since derivatives with respect to ω_2 cancel). Equation (2.43) can also be expressed in the form

$$x - x_1 = \left(\omega_2^0 - \frac{\beta_1}{\frac{\partial \beta_1}{\partial \omega_2^0}} \right) t,$$

$$y = \frac{t}{\frac{\partial \beta_1}{\partial \omega_2^0}}, \tag{2.46}$$

where $x_1 = -f - x$ from (2.22), and $\omega_2' = \omega_2 - \omega_2^0$ is small. One can then easily obtain $x_2 = -\omega_2 \tau$ from (2.43) and (2.46). Expression (2.22) takes the form

$$P = \frac{f(\omega_2)}{\sqrt{\omega_2}} \sqrt{-\frac{x_1}{t}}, \quad P = f(\omega_2) \sqrt{\frac{\tau}{t}}, \tag{2.47}$$

where

$$\frac{f(\omega_2)}{\sqrt{\omega_2}} = -\frac{P_1^0 V}{\pi} Z(\omega_{1s})\, a_0^2 \sqrt{\frac{1}{|f''(\omega_{1s})|}}\, 2\sqrt{2}\, \frac{1}{-\omega_{1s}^2 + V^2} \sqrt{t},$$

$$\sqrt{\frac{t}{y\,|f''(\omega_{1s})|}} = \sqrt{\frac{\frac{\partial \beta_1}{\partial \omega_2}}{\frac{\partial^2 \beta_1}{\partial \omega_2^2}}}.$$

Denote $\frac{c_1 + u_n}{P} = A(\theta_0, c_0, a_0, a_1)$, where c_1 is taken for $\theta' = 0$. If the linear characteristic variable τ is replaced by the improved one τ_1 in solution (2.47), then the characteristics are given by

$$\tau_1 = \text{const}, \quad -c_0 \frac{\partial \tau}{\partial t} = A \frac{f(\omega_2)}{Bn} \sqrt{\frac{\tau_1}{t}}. \tag{2.48}$$

After integration, the solution takes the form

$$P = f(\omega_2) \sqrt{\frac{\tau_1}{t}}, \quad \tau = -\frac{A f(\omega_2)}{c_0 Bn}\, 2\sqrt{\tau_1}\, \sqrt{t} + \tau_1,$$

$$x = \left(\omega_2 - \frac{\beta_1}{\frac{\partial \beta_1}{\partial \omega_2}} \right)(t - \tau), \quad y = \frac{t - \tau}{\frac{\partial \beta_1}{\partial \omega_2}}.$$

This solution satisfies the equation $\frac{\partial P}{\partial \tau}\left(\tau + \frac{AP}{C_0 Bn} \right) - \frac{1}{2} P = 0$ which is one-dimensional in τ. This equation is derived from the nonlinear equations of motion by transforming from the variables x, y to τ, ω_2, linearizing terms containing derivatives with respect to ω_2, and expressing all the parameters in terms of P from the compatability conditions on the characteristic. Relations (2.47) and (2.48) hold near

the wave BB' of Figure 5. The shock wave is determined by the following relations which are satisfied on it:

$$(u_n - U)\rho = -U\rho_0,\ (u_n - U)\rho u_n + P + \frac{\mu_e H^2}{2} - \mu_e \frac{B^{\circ 2}}{2} = 0,$$

$$H_n = B^\circ \cos\theta,$$

$$(u_n - U)\rho u_\tau - \mu_e H_n H_\tau = -\mu_e H_n B^\circ \sin\theta;$$

τ is tangential to BB' and n is the normal; $\frac{\rho}{\rho_0} = \left(1 + \frac{P}{B}\right)^{\frac{1}{n}}$.

The last equation agrees to second order with the energy equation /53/. Hence the shock velocity U is given by $\theta = \theta_0 + \theta_1'$, $U = c_0 + U_1$, where

$$U_1 = \frac{a_0^2}{2c_0} \frac{\frac{n+1}{2}\frac{c_0^2}{a_0^2}(a_1^2 - c_0^2) + \frac{c_0^2 a_1^2 \sin^2\theta_0}{a_1^2\cos^2\theta_0 - c_0^2}}{(a_1^2\cos^2\theta_0 - c_0^2)^2 + a_1^4 \sin^2\theta_0 \cos^2\theta_0} (a_1^2\cos^2\theta_0 - c_0^2)\frac{P}{Bn} -$$

$$- \frac{c_0^2}{2c_0} \frac{(a_1^2 - c_0^2)\, a_1^2 \theta_1'\, 2\sin\theta_0\cos\theta_0}{(a_1^2\cos^2\theta_0 - c_0^2)^2 + a_1^4\sin^2\theta_0\cos^2\theta_0}.$$

Relation (2.44) holds on the shock wave, where on the right-hand side we have U' and $\frac{\partial\tau}{\partial t} = \frac{\tau}{t}$. By virtue of its one-dimensionality in r and t, the term containing θ_1 in U' cancels with the term containing $\frac{\partial\tau}{\partial\omega_2}$ in (2.44). To determine τ_1 on the shock front we have (2.47) and $-\frac{\tau}{t} = \frac{U_1}{c_0}$, where in U_1 we set $\theta_1' = 0$ and $\tau = \tau_1$; $\sqrt{\tau_1}$ is found in finite form. Owing to the fact that U_1 is cumbersome, only the axisymmetric case was considered in detail, for which $x = 0$, $\cos\theta_0 = 0$; hence

$$c_1 + u_n = \frac{n+1}{2}\frac{a_0^2}{c_0}\frac{P}{Bn} + \frac{3}{2}\frac{a_1^2}{c_0}\frac{P}{Bn},\ U = \frac{a_0^2}{c_0}\left(1 - \frac{n+1}{4}\frac{P}{Bn}\right) +$$

$$+ \frac{a_1^2}{c_0}\left(1 + \frac{n+1}{4}\frac{P}{Bn} + \frac{2-n}{4}\frac{P}{2n}\right).$$

In the present case $D = \frac{c + u_n + c_1}{2}$, which is also true in general. Therefore, we have from (2.48)

$$-A\frac{f(\omega_2)}{c_0 Bn} 2\sqrt{\frac{\tau_1}{t}} + \frac{\tau_1}{t} = -\frac{A}{2}\frac{f(\omega_2)}{c_0 Bn}\sqrt{\frac{\tau_1}{t}};\ \sqrt{\frac{\tau_1}{t}} = \frac{A f(\omega_2)}{c_0 Bn}$$

on the shock front, where

$$A = \frac{n+1}{2}\frac{a_0^2}{c_0} + \frac{3}{2}\frac{a_1^2}{c_0} \text{ on } Oy.$$

3. The previous problem is now considered in the presence of a vertical field

$$B_x = 0,\ B_y = B^\circ.$$

The equations of motion in the upper medium are, as before, independent, and for the lower medium have the form /49/

$$\tau b_x = B^\circ \xi_2 v_x,$$
$$\tau b_y = -B^\circ \xi_1 v_x,$$
$$\tau v_x = -\frac{1}{\rho_0}\xi_1 P - \frac{a_1^2}{B_0}(\xi_1 b_y - \xi_2 b_x),$$
$$\tau v_y = -\frac{1}{\rho_0}\xi_2 P,$$
$$\tau P + \rho_0 a_0^2 \xi_1 v_x + \rho_0 a_0^2 \xi_2 v_y = 0.$$

Eliminating the unknowns, the pressure is found to obey the equation

$$((\tau^2 - a_1^2\xi_1^2 - a_1^2\xi_2^2)(\tau^2 - a_2^2\xi_2^2) - a_0^2\xi_1^2\tau^2)P = 0. \tag{3.1}$$

The solution of (3.1) is sought in the form

$$P = e^{ikx + i\beta y + st}$$

which gives rise to a relation between k, β and s, i.e., the surface defined by the normals:

$$(s^2 + a_1^2(k^2 + \beta^2))(s^2 + a_0^2\beta^2) + s^2 a_0^2 k^2 = 0 \tag{3.2}$$

or

$$\beta_{1,2}^2 = \frac{-(s^2a_0^2 + s^2a_1^2 + a_0^2a_1^2k^2) \pm}{2a_1^2a_0^2} \times$$
$$\times \frac{\sqrt{(s^2a_0^2 + s^2a_1^2 + a_0^2a_1^2k^2)^2 - 4a_1^2a_0^2(s^4 + a_1^2k^2 + s^2a_0^2k^2)}}{2a_1^2a_0^2}. \tag{3.3}$$

In contrast to § 2, there are here two values of β^2, i.e., the number of boundary conditions should be larger. The wave fronts are similar to those in Figure 19, except that the x axis is replaced by the y axis. The boundary conditions on $y = 0$ give the following equalities for the pressure and magnetic field components /53/:

$$P = P_1, \quad b_x = b_{x_1}, \quad b_y = b_{y_1}. \tag{3.4}$$

Introducing transforms for P, b_x and b_y, we obtain, e.g. for P,

$$\overline{P} = \int_{-\infty}^{\infty} e^{-ikx}\,dk \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{st}\,ds\,(A_1 e^{i\beta_1 y} + A_2 e^{i\beta_2 y}). \tag{3.5}$$

For b_x and b_y we have

$$\overline{b}_x = A_1 e^{i\beta_1 y}\frac{\frac{B^\circ}{\rho_0}k\beta_1}{s^2 + a_1^2k^2 + a_1^2\beta_1^2} + A_2 e^{i\beta_2 y}\frac{\frac{B^\circ}{\rho_0}k\beta_2}{s^2 + a_1^2k^2 + a_1^2\beta_2^2},$$

$$\bar{b}_y = A_1 e^{i\beta_1 y} \frac{-\frac{B^1}{\rho_0} k^2}{s^2 + a_1^2 k^2 + a_1^2 \beta_1^2} + A_2 e^{i\beta_2 y} \frac{-\frac{B^2}{\rho_0} k^2}{s^2 + a_1^2 k^2 + a_1^2 \beta_2^2}, \tag{3.6}$$

and b_{x_1}, b_{y_1} have the representations of § 2. The conditions $\bar{b}_x = \bar{b}_{x_1}$, $\bar{b}_y = \bar{b}_{y_1}$, $\bar{p} = \bar{p}_1$ on $y = 0$ yield

$$A_2 = A_1 \frac{-ik - \beta_1}{\beta_2 + ik} \frac{s^2 + a_1^2 k^2 + a_1^2 \beta_2^2}{s^2 + a_1^2 k^2 + a_1^2 \beta_1^2},$$

$$A_1 + A_2 = \frac{1}{2\pi} \int_0^\infty e^{-st'} dt' \int_{-\infty}^{\infty} e^{ikx} P_1 \, dx'. \tag{3.7}$$

Substituting (3.7) in (3.5), a value for P is obtained which, near the fronts, can be investigated by the methods of § 2.

Now consider the axisymmetric problem. In this case equation (3.1) becomes

$$\left(\left(\tau^2 - a_1^2 \xi_1^2 - a_1^2 \xi_1 \frac{1}{x} - a_1^2 \xi_2^2\right)(\tau^2 - a_0^2 \xi_2^2) - a_0^2 \left(\xi_1^2 + \frac{\xi_1}{x}\right)\right) P = 0. \tag{3.8}$$

A solution of (3.8) is sought in the form

$$P = e^{st} e^{i\beta y} A(x), \tag{3.9}$$

where $A(x)$ satisfies

$$\frac{d^2 A}{dx^2} + \frac{1}{x} \frac{dA}{dx} + \nu^2 A = 0,$$

$$\nu^2 = -\frac{(s^2 + a_1^2 \beta^2)(s^2 + a_0^2 \beta^2)}{a_1^2 s^2 + a_0^2 s^2 + a_0^2 a_1^2 \beta^2}, \tag{3.10}$$

whose solution is

$$A(x) = J_0(\nu x), \quad \nu = k. \tag{3.11}$$

The pressure in (3.5) is replaced by

$$P = \int_{c-i\infty}^{c+i\infty} \frac{1}{2\pi i} e^{st} ds \int_0^\infty J_0(kx) \, k dk \, (A_1 e^{i\beta_1 y} + A_2 e^{i\beta_2 y}), \tag{3.12}$$

where $\beta_{1,2}$ is given in (3.3). For A_1 and A_2 we have equations (3.7), and

$$A_1 = \gamma(s, k) \int_0^\infty e^{-st'} dt' \int_0^\infty J_0(kx') \, P_1 \, x' \, dx',$$

where

$$\operatorname{Re}\frac{1}{1+\dfrac{-ik-\beta_1}{\beta_2+ik}\dfrac{s^2+a_1^2k^2+a_1^2\beta_2^2}{s^2+a_1^2k^2+a_1^2\beta_1^2}}=\gamma(s,k) \tag{3.13}$$

is found from (3.7). The asymptotic behavior of (3.11) for large values of kx on the front AB is found from the asymptotic representations of the Bessel function /52/, i.e.,

$$J_0(kx)=\left(\frac{2}{\pi kx}\right)^{\frac{1}{2}}\cos\left(kx-\frac{\pi}{4}\right),$$

so that for large values of kx and kx' near the front AB

$$J_0(kx)\,J_0(kx')=\frac{1}{\pi k\sqrt{xx'}}\cos k(x-x'),$$

and for the pressure on the front AB

$$P=\sqrt{\frac{x_s'}{x}}P_1\gamma. \tag{3.14}$$

If the axisymmetric problem on the wave front AB is solved in the absence of acceleration, we can find for $R'(t)=V$

$$\frac{P}{\gamma P_1}=\sqrt{\frac{\eta_s}{x}},\ \eta_s=x_s', \tag{3.15}$$

i.e., the known law in the axisymmetric case.

The system of equations (2.3) can be satisfied by introducing a function $\psi(x, y, t)$, where $b_x=-B^\circ a_0\dfrac{\partial^3\psi}{\partial y^2\partial t}$, $b_y=B^\circ a_0\dfrac{\partial^3\psi}{\partial x\partial y\partial t}$, $\dfrac{P}{\rho_0 a_0}=-\dfrac{\partial^3\psi}{\partial t^3}+a_1^2\left(\dfrac{\partial^3\psi}{\partial t\partial x^2}+\dfrac{\partial^3\psi}{\partial t\partial y^2}\right)$, $\dfrac{v_x}{a_0}=\dfrac{\partial^3\psi}{\partial x\partial t^2}-a_1^2\left(\dfrac{\partial^3\psi}{\partial x^3}+\dfrac{\partial^3\psi}{\partial y^2\partial x}\right)$, $\dfrac{v_y}{a_0}=\dfrac{\partial^3\psi}{\partial y\partial t^2}$. System (2.2) will also be satisfied, the function ψ obeying equation (2.4), which is derived from the last equation of this system. With the introduction of a function φ defined by

$$\frac{\partial\psi}{\partial t}=\varphi, \tag{3.16}$$

(2.4) will also hold for φ, where

$$b_x=-B^\circ a_0\frac{\partial^2\varphi}{\partial y^2},\ b_y=B^\circ a_0\frac{\partial^2\varphi}{\partial x\partial y},\ \frac{P}{\rho_0 a_0}=-\frac{\partial^2\varphi}{\partial t^2}+a_1^2\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right). \tag{3.17}$$

Consider P_1 on the boundary $y=0$, and at the point $x=0$, $t=0$; the function φ will then be a homogeneous function of zero dimension. It can be easily

shown that the function

$$\varphi = \operatorname{Re}\varphi(k_1) \tag{3.18}$$

satisfies equation (2.4), where k_1 is found from the plane wave equation

$$k_1 x + \beta_1(k_1)\, y = t; \tag{3.19}$$

$\beta_1(k_1)$ is given by (2.7), and $\varphi(k)$ is an analytic function. With the aid of (3.17) we find that on $y=0$ $(k_1=k)$

$$\frac{b_x}{B^\circ a_0} = -\operatorname{Re}\left(\varphi''(k_1)\left(\frac{\beta_1}{x}\right)^2 + \varphi'(k_1)\frac{2\beta_1'\beta_1}{x^2}\right);$$

$$\frac{b_y}{B^\circ a_0} = \operatorname{Re}\left(\varphi''\frac{k_1\beta_1}{x^2} - \varphi'\frac{\beta_1 + k_1\beta_1'}{x^2}\right).$$

$$\frac{P}{\rho_0 a_0} = \operatorname{Re}\left(\varphi''\frac{1}{x^2}(k_1^2 a_1^2 - \beta_1^2 a_1^2 - 1) + \varphi'\frac{2a_1^2}{x^2}(k_1 + \beta_1\beta_1')\right). \tag{3.20}$$

In the upper half-space

$$\frac{b_{x_1}}{B^\circ} = \frac{\partial\varphi_1}{\partial x},\quad \frac{b_{y_1}}{B^\circ} = \frac{\partial\varphi_1}{\partial y},\quad \varphi_1 = \frac{1}{t}\operatorname{Re}\varphi_1(k_0),$$

$$k_0 x - ik_0 y = t; \tag{3.21}$$

hence, on $y = 0$, $k_0 = k$:

$$\frac{b_{x_1}}{B_0} = -\frac{1}{t}\operatorname{Re}\varphi_1'(k)\frac{k}{x},\quad \frac{b_{y_1}}{B^\circ} = \frac{1}{t}\operatorname{Re}\varphi_1'(k)\frac{ik}{x}. \tag{3.22}$$

Substituting (3.20) and (3.22) in (2.1), a system is derived for the functions $\varphi(k)$, $\varphi_1(k)$. If $P_1 = 0$ at the boundary,

$$\varphi''(k^2 a_1^2 - 1) + \varphi' \cdot 2ka_1^2 = -\frac{\varphi_1'}{a_0}a_1^2 + c_1 i,$$

$$\varphi'' k\beta_1 + \varphi'(\beta_1 + k\beta_1') = \varphi_1' i + c_2 i, \tag{3.23}$$

where one can set $c_1 = 0$, $c_2 = 0$ from the conditions on the axis $x = 0$. When φ_1' is eliminated, (3.23) yields

$$\varphi''(1 - k^2 a_1^2 - ik\beta_1 a_1^2) = \varphi_1' a_1^2(2k + i\beta_1 + k\beta_1').$$

The solution of this equation is

$$\varphi' = Cie^{\int \frac{a_1^2(2k + i\beta_1 + ik\beta_1')}{1 - k^2 a_1^2 - ik\beta_1 a_1^2}dk},\quad \varphi' = \frac{Ci}{1 - k^2 a_1^2 - ik\beta_1 a_1^2}, \tag{3.24}$$

where the constant C can be expressed in terms of the momentum (/27/, p. 96)

$$C = \frac{P_1}{2\pi\rho_0 a_0}.$$

For a pressure distribution $P_1(x,t)$ on the surface, the solution $\varphi(x,j,t)$ can be found in terms of $\varphi(k_1)$ due to a momentum applied at the point (x',t'), where the relation

$$k_1(x-x')+\beta_1 y = t-t' \tag{3.25}$$

is obtained by integration over the (x',t')-plane (/27/, p. 98). The pressure near the front BB' is determined by (2.22).

The pressure has the form

$$\frac{P}{\rho_0 a_0} = \mathrm{Re}\left\{\int\int_G \frac{\varphi'\beta_1^* y\,(1-a_1^2k_1^2-a_1^2\beta_1^2)}{(x-x'+\beta_1' y)^3}dx'dt' - \int\int_G \frac{(\varphi'(1-a_1^2k_1^2-a_1^2\beta_1^2))'}{(x-x'+\beta_1' y)^2}\,dx'dt'\right\},$$

where the region G is bounded by

$$k_1(x-x')+\beta_1 y = t-t',\ x-x'+\beta_1' y = 0,$$
$$x' = R(t').$$

Near the wave front AB, after the substitutions $x-x'+\beta_1'(k_1)y=\eta$, $\eta'=x-x'+\beta_1'(k_0)y$, $t-t'=(\beta_1-\beta_1'k_0)y$, where $k=k_0+k'$, k' being small, we obtain

$$k'\eta = -2\eta' k_0,\ \eta = \sqrt{-2\eta' k_0 y\beta_1^*},\ \frac{\partial t'}{\partial \eta'} = k_0,\ \sqrt{-\eta'} = \eta'',\ \frac{\partial \eta'}{\partial y} = \frac{\beta_1}{k_0},$$

$$\frac{P}{\rho_0 a_0} = \mathrm{Re}\,\frac{\partial}{\partial y}\int_0^{\eta_1}\int_{r_2^*}^{r_1^*} \frac{4\varphi'(1-a_1^2k_0^2-a_1^2\beta_1^2)\,k_0\,dx'd\eta''}{\sqrt{2\beta_1^* k_0}\sqrt{y}},$$

retaining small k'^2 where η_1 is the value of $\sqrt{-\eta'}$ for $x'=0$ and $t'=0$, $|r_{1,2}|=R(t')$, and $y=t/\beta_1-k_0\beta_1'$ near AB. If the function dependent on k_0 is taken outside the integral, then for large t the following is valid: $t'=\tau-k_0 r_{1,2}^*$, $\eta_1=\sqrt{\tau}$,

$$P = \frac{F(\tau)}{\sqrt{t}}.$$

The nonlinear solution on AB can now be found from (2.48) with the substitution $\tau = \tau_1$.

4. We now consider the motion of a viscous fluid due to pressure on its surface. The equation for small motions is /56/

$$\frac{\partial^2 P}{\partial t^2} = \left(a_0^2 + \frac{4}{3}\nu\frac{\partial}{\partial t}\right)\Delta P, \tag{4.1}$$

with the boundary condition, as before, given by (1.1). The problem is

assumed to have axial symmetry. A Laplace transform for P gives (4.1) the form

$$\Delta\overline{P} - \frac{s^2}{a_0^2 + \frac{4}{3}\nu s}\,\overline{P} = 0, \tag{4.2}$$

the boundary condition $\overline{P}_1$ being derived from (1.1).

The solution of (4.2) subject to the boundary condition $\overline{P}(x, 0) = \overline{P}_1$ is (/27/, p. 142)

$$\overline{P} = -\frac{1}{2\pi}\frac{\partial}{\partial y}\int\int \frac{\overline{P}_1\, e^{-\frac{s}{\sqrt{a_0^2 + \frac{4}{3}\nu s}}R}}{R}\, r dr d\psi, \tag{4.3}$$

where

$$R = \sqrt{x^2 + y^2 + r^2 - 2xr\cos\psi}$$

and the region of the integration extends over the (r, ψ)-plane $(y = 0)$. For large times t, a considerable contribution is made to the solution (4.3) by values $s \approx 0$ /57/. The expression $st - \frac{sR}{\sqrt{a_0^2 + \frac{4}{3}\nu s}}$ appears in the exponent for P, and so for $s \approx 0$ a stationary point $a_0 t = R$ exists. For small s

$$\frac{s}{\sqrt{a_0^2 + \frac{4}{3}\nu s}} = \frac{s}{a_0} - \frac{1}{2}\,\frac{4}{3}\,\frac{\nu s^2}{a_0^3}; \tag{4.4}$$

hence

$$\overline{P} = -\frac{1}{2\pi}\frac{\partial}{\partial y}\int\int \frac{\overline{P}_1 e^{-\frac{s}{a_0}R}\; e^{\frac{1}{2}\frac{4}{3}\frac{\nu s^2}{a_0^3}R}}{R} - r dr d\psi, \tag{4.5}$$

for which the convolution theorem yields

$$P = -\frac{1}{2\pi}\frac{\partial}{\partial y}\int\int_S \frac{1}{\sqrt{4\pi\,\frac{4}{6}\,\frac{\nu}{a_0^3}R}} \times$$

$$\times \frac{\int_0^t P_1(r, t')\, e^{-\frac{\left(t - t' - \frac{R}{a_0}\right)^2}{4\,\frac{4\nu}{6a_0^3}R}}\, dt'}{R}\, r dr d\psi,$$

where the region of integration is bounded by $r \leqslant R(t)$. When $\nu = 0$ a δ-function appears in the inner integral corresponding to a front, ahead

of which $P=0$. When $\nu>0$ there is diffusion and $P\neq 0$ everywhere. However /57/, if $\int_0^\infty P_1\,dt'$ and $R(\infty)$ are finite (§ 6, Chapter II), the main part of the disturbances as $t\to\infty$ are concentrated in the vicinity of the sound wave $R=a_0t$, which can be easily found by the stationary-phase method. The pressure for large t is given by

$$P=\frac{1}{2\pi}\frac{\sin\theta}{\sqrt{4\pi\frac{4}{6}\frac{\nu}{a_0^3}R}}\frac{1}{R}\,2\,\frac{t-\frac{R}{a_0}}{4\frac{4\nu}{6a_0^3}R}\,e^{-\frac{(t-R/a_0)^2}{4\frac{4\nu}{6a_0^3}R}}\iint_s rdrd\psi\int_0^\infty P_1(r,\,t')\,dt'. \quad (4.6)$$

In the one-dimensional case the form of (4.6) resembles the attenuation law of a shock wave in an inviscid fluid /13/. In the three-dimensional case, however, such a similarity does not exist: the attenuation law in (4.6) varies with time according to $P\sim\frac{1}{t^{3/2}}\left(t-\frac{R}{a_0}\right)$, and does not agree with the case of an inviscid fluid for which $P\sim 1/t\sqrt{\ln t}$.

The equations for a viscous fluid in the vicinity of the sound wave are used to examine the above similarity in more detail. In the case of one-dimensional motion, e.g., the problem of a piston moving with velocity $f(t)$, where $f(t)$ drops to zero at $t=t_0$, the equations accounting for viscosity become

$$\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{4}{3}\nu\frac{\partial^2 v_x}{\partial x^2},$$

$$\frac{\partial\rho}{\partial t}+v_x\frac{\partial\rho}{\partial x}+\rho\frac{\partial v_x}{\partial x}=0,$$

$$\frac{dP}{d\rho}=a^2.$$

In the following $\frac{f}{a_0}$ is assumed small. The adiabatic relation $\frac{dP}{d\rho}=a^2$ is used; the energy equation will not be written. The reason is that in the equation $dP=a^2d\rho+\left(\frac{\partial P}{\partial s}\right)_\rho dS$ the entropy is ignored in virtue of the energy equation, if only viscosity is taken into account. Using the short-wave approximation /45/, the solution is obtained near the shock wave for large times. With the notation

$$\frac{x-a_0t}{\frac{n+1}{2}a_0t}=\delta,\quad\frac{v_x}{a_0}=\mu,\quad\frac{P}{\rho_0a_0^2}=\alpha,$$

and discarding small quantities of order higher than first, one can derive the following in terms of the variables δ and t:

$$\alpha=\mu,\quad t\frac{\partial\mu}{\partial t}-(\mu-\delta)\frac{\partial\mu}{\partial\delta}-\frac{2\nu}{3t}\frac{\partial^2\mu}{\partial\delta^2}\frac{1}{\left(\frac{n+1}{2}\right)^2}=0. \quad (4.7)$$

If viscosity forces are disregarded, the solution of this equation is

$$\mu = \delta - \frac{\Phi(\mu)}{t}.$$

The arbitrary function Φ is found by the technique of improving the characteristics /4/. In the solution $\mu = f\left(t - \frac{x}{a_0}\right)$ of the linear problem, the quantity $t - \frac{x}{a_0}$ is replaced by y', where $y' = \text{const}$ is the equation of the characteristics:

$$\mu = f(y'),\ t = \frac{x}{a_0} - \frac{n+1}{2}\frac{f(y')}{a_0}x + y',\ t(\delta - \mu)\frac{n+1}{2} = -y'. \tag{4.8}$$

For large values of t, the orders of the terms in equation (4.7), namely

$$f'(\delta - \mu),\ f'(\delta - \mu),\ \frac{\nu}{t}\frac{f''}{f'^2}, \tag{4.9}$$

can be found quite easily. From the conditions on the shock front /4/, we have (§ 6, Chapter II)

$$f(y') = \sqrt{\frac{2\int_0^{y_0} f(y')\,dy'}{\frac{n+1}{2}t}}, \tag{4.10}$$

where y_0 satisfies $f(y_0) = 0$. Now, since in virtue of (4.8) $\delta - \mu = -\frac{y_0}{\frac{n+1}{2}t}$, the ratio of the viscous term in (4.9) to the remaining term, $\frac{\nu f''}{f'^3} \ll 1$, since usually (§ 4, Chapter I) $\nu \ll f'^2$. Thus, in one-dimensional motion the effect of the nonlinearity in solution (4.10) exceeds that of the viscosity. The same result can be obtained by discarding the nonlinearity in (4.7) and solving the linear problem for a viscous fluid. For large times, we then have in the vicinity of the sound wave /57/

$$v_x = \frac{1}{\sqrt{\frac{8\pi}{3}\frac{\nu x}{a_0^3}}}\int_0^t f(\tau)\, e^{-\frac{(t-\tau-x/a_0)^2}{\frac{8}{3}\frac{\nu x}{a_0^3}}}\,d\tau,$$

$$v_x = e^{-\frac{8\pi t}{3}\frac{\nu t}{a_0^2}\left(\frac{n+1}{2}\right)^2}\frac{1}{\sqrt{\frac{8\pi}{3}\frac{\nu t}{a_0^2}}}\int_0^{t_0} f(\tau)\,d\tau. \tag{4.11}$$

Now since the order of δ is $\sqrt{\frac{\nu}{t}}$, in virtue of (4.11) the following orders of magnitude hold:

$$\mu = \frac{f}{\sqrt{\nu t}},\ \frac{\partial\mu}{\partial t} = \frac{f}{\sqrt{\nu t}}\frac{\delta^2}{\nu},\ \frac{\partial\mu}{\partial\delta} = \frac{f}{\sqrt{\nu t}}\frac{\delta t}{\nu},\ \frac{\partial^2\mu}{\partial\delta^2} = \frac{f}{\sqrt{\nu t}}\left(\frac{\delta t}{\nu}\right)^2.$$

Multiplying out the parentheses in (4.7), we see that the first, third and fourth terms are of the order of $f t\delta^2/\nu\sqrt{\nu t}$, whereas the second term $\mu\frac{\partial\mu}{\partial\delta}$, which represents the allowance for the nonlinearity, is of the order of $\frac{f^2}{\sqrt{\nu t}}\frac{\delta t}{\nu\sqrt{\nu t}}$. Since $\delta=\sqrt{\frac{\nu}{t}}\cdot\sqrt{\frac{\nu}{t}}\ll\sqrt{\frac{f}{t}}$, the nonlinearity is again seen to be more important than the viscosity, so that (4.11) is incorrect as $t\to\infty$. The cases of cylindrical and spherical symmetry correspond to solutions (6.19) and (6.9) of Chapter II. In the first case, i.e., for the two-dimensional problem, an equation for μ exists /45/; after simplifications and taking into account (in virtue of solution (6.19)) that the motion is one-dimensional, we have

$$t\frac{\partial\mu}{\partial t}+(\mu-\delta)\frac{\partial\mu}{\partial\delta}+\frac{1}{2}\mu\frac{2\nu}{3t}\frac{\partial^2\mu}{\partial\delta^2}\frac{1}{\left(\frac{n+1}{2}\right)^2}=0. \tag{4.12}$$

(Incidentally, as $t\to\infty$, (4.6) also gives one-dimensional motion in R.) If $R_1=r-a_0t$, $\left.\frac{\partial\mu}{\partial t}\right|_\delta=\left.\frac{\partial\mu}{\partial t}\right|_{R_1}+\frac{\partial\mu}{\partial\delta}\frac{\delta}{t}$, it is easily seen that (4.6) satisfies (4.12), and if the nonlinear term in (4.12) is discarded, (4.11) also satisfies the linearized equation (4.7). When viscosity is neglected for large t solution (6.19) of Chapter II is derived.

For large t

$$\mu=\frac{F}{\sqrt{t}},\ \frac{\partial\mu}{\partial t}=\frac{F}{t\sqrt{t}},\ \frac{\partial\mu}{\partial\delta}=1,\ \frac{\partial^2\mu}{\partial\delta^2}=\frac{F''}{F'^2}\sqrt{t}.$$

The first three terms in (4.12) are of the order of $\frac{F}{\sqrt{t}}$, and the last term is of the order of $-\frac{\nu}{2\sqrt{t}}\frac{F''}{F'^2}$. In virtue of the conditions on the shock wave $F=\frac{1}{\sqrt{t}}$, and for $\sqrt{t}\,\nu\gg1$ the viscosity effect exceeds that of the nonlinearity. Thus, terms giving the largest attenuation remain in the equations. In the axisymmetric case, according to (6.9) of Chapter II,

$$\frac{\partial\mu}{\partial t}=\frac{F}{t^2},\ \frac{\partial\mu}{\partial\delta}=\frac{1}{\ln t},\ \frac{\partial^2\mu}{\partial\delta^2}=\frac{F''t}{(F'\ln t)^2}$$

and the orders of the terms in equation (4.12) ($\frac{1}{2}\mu$ is replaced by μ) are given by

$$\frac{F}{t},\ \left(\frac{F}{t}-\frac{F\ln t}{t}\right)\frac{1}{\ln t},\ \frac{F}{t},\ -\frac{\nu t F''}{2F'^2\ln^2 t},$$

where $F=\frac{1}{\sqrt{\ln t}}$. Here, the effect of viscosity much exceeds that of the nonlinearity. The linear solution (4.6) for a viscous fluid has the following

orders:

$$\mu \quad \frac{1}{t\sqrt{\nu t}}\frac{\delta}{\nu},\ \frac{\partial\mu}{\partial t}=\frac{\delta^3}{t\sqrt{\nu t}\,\nu^2},\ \frac{\partial\mu}{\partial\delta}=\frac{\delta^2}{\nu^2\sqrt{\nu t}},$$
$$\frac{\partial^2\mu}{\partial\delta^2}=\frac{\delta^3 t}{\nu^3\sqrt{\nu t}},\ \delta \quad \sqrt{\frac{\nu}{t}};$$

the effect of the nonlinearity $\mu\frac{\partial\mu}{\partial\delta}=\frac{\delta^3}{\nu^4t^2}$ in (4.12) is then much smaller than the remaining terms, having an order of $\frac{\delta^3 t}{\nu^2\sqrt{\nu t}\,t}$, since $(\sqrt{t})^3\nu^{\frac{3}{2}}$ 1, νt 1. If T denotes the attenuation time of the pressure at the surface, then according to (4.6) the viscosity is important for times $\sqrt{\frac{\nu t}{a_0^2}}$ T; for water $t\sim 10^6\,T$.

For times when both effects should be taken into account, the one-dimensional problem can be solved making allowance both for the viscous and nonlinear terms /45/. There is an inaccuracy in /45/, in that one should write $\partial/\partial x$ in place of $\partial/\partial z$ in the shortened equations there. With the notation

$$R_1=tb,\ -R_1\frac{n+1}{2}=\tau\ a_0^{-1},\ \alpha=\frac{n+1}{2},\ \delta_1=\frac{2\nu}{3a_0},\ \mu=v,$$

equation (4.7) becomes

$$\frac{\partial v}{\partial x}-\alpha v\frac{\partial v}{\partial\tau}=\delta_1\frac{\partial^2 v}{\partial\tau^2}$$

in terms of the variables x and τ. The substitution /45/

$$v=\frac{2\delta_1}{\alpha}\frac{\frac{\partial U}{\partial\tau}}{U}$$

linearizes the indicated equation, which becomes

$$\frac{\partial U}{\partial x}=\delta_1\frac{\partial^2 U}{\partial\tau^2}$$

subject to the conditions

$$U(x,0)=1,\ U(0,\tau)=e^{\int_0^\tau\frac{\alpha}{2\delta_1}v(0,t)\,dt}$$

The solution of the problem takes the form

$$U(x,\tau)=\frac{1}{2}\frac{1}{\sqrt{\pi\delta_1 x}}\int_0^\infty e^{-\frac{(\xi-\tau)^2}{4\delta_1 x}}U(0,\xi)\,d\xi+\frac{2}{\sqrt{\pi}}\int_{\frac{\tau}{2\sqrt{\delta_1 x}}}^\infty e^{-u^2}du.$$

As $\delta_1\to 0$, i.e., when passing to an ideal fluid for large t, the stationary-

phase method may be applied to the integral and the stationary point

$$\xi = \tau + \alpha\, vx$$

obtained for the solution

$$v = v\,(0, \xi),$$

which agrees with (4.8). In the case of cylindrical and spherical waves, the solution is not obtained in simple form. Here, numerical methods or the method of matching ideal and stationary solutions can be used /45/. Allowance for the dispersion effect of an inhomogeneous fluid is made in /57, 63/.

BIBLIOGRAPHY

1. Sedov, L. I. O nekotorykh neustanovivshikhsya dvizheniyakh szhimaemoi zhidkosti (Unsteady Motions of Compressible Fluids).— PMM, Vol. 9, No. 4. 1945.
2. Khristianovich, S. A. Udarnaya volna na bol'shikh rasstoyaniyakh ot mesta vzryva (Shock Wave at Considerable Distance from the Place of Explosion).— PMM, Vol. 20, No. 5. 1956.
3. Lighthill, M. J. The Shock Strength in Supersonic "Conical Fields".— Philos. Mag., Ser. 7, Vol. 40, p. 1202. 1949.
4. Whitham, G. B. The Propagation of Weak Spherical Shocks in Stars.— Comm. Pure Appl. Math., Vol. 6, p. 397. 1953.
5. Whitham, G. B. The Propagation of Spherical Blast.— Proc. Roy. Soc., Ser. A, Vol. 203, p. 571. 1950.
6. Grib, A. A., A. G. Ryabinin, and S. A. Khristianovich. Ob otrazhenii ploskoi udarnoi volny v vode ot svobodnoi poverkhnosti (On the Reflection of a Plane Shock Wave in Water from a Free Surface).— PMM, Vol. 20, No. 2. 1956.
7. Yanenko, N. N. Begushchie volny sistemy kvazilineinykh uravnenii (Traveling Waves of a System of Quasilinear Equations).— DAN SSSR, No. 109. 1956.
8. Legras, M. J. Nouvelles Applications de la Méthode de Lighthill à l'etude des ondes de choc.— O. N. E. R. A. Publ. No. 66, Paris. 1953.
9. Grib, A. A., O. S. Ryzhov, and S. A. Khristianovich. Teoriya korotkikh voln (Theory of Short Waves).— PMTF, No. 1, 1960; PMM, Vol. 22, No. 5. 1958.
10. Whitham, G. B. A New Approach to Problems of Shock Dynamics Part I: Two-dimensional Problems.— J. Fluid Mech., Vol. 2, p. 145. 1957.
11. Dryden, H. L. and T. Kármán, ed. Advances in Applied Mechanics, New York— London, Academic Press.

    Tsien, Hsue-Shen. The Poincaré—Lighthill—Kuo Method, Vol. 4, p. 281, 1956;

    Frieman, E. A. and R. M. Kulsrud. Problems in Hydromagnetics, Vol. 5, p. 195, 1958;

    Chester, W. The Propagation of Shock Waves along Ducts of Varying Cross Section, Vol. 6, p. 119, 1960.

12. Stanyukovich, K.P. Neustanovivshiesya dvizheniya sploshnoi sredy (Unsteady Motions of a Continuous Medium).— Gostekhizdat. 1955.
13. Landau, L.D. and E.M. Lifshits. Mekhanika sploshnykh sred (Mechanics of Continuous Media).— Moskva, Gostekhizdat. 1953.
14. Sedov, L.I. Similarity and Dimensional Methods in Mechanics.— London, Cleaver—Hume Press. 1959.
15. Courant, R. and K.O. Friedrichs. Supersonic Flow and Shock Waves.— New York, Interscience. 1948.
16. Frankl', F.I. and E.A. Karpovich. Gas Dynamics of Thin Bodies.— New York, Interscience. 1953.
17. Saromonyan, A.Ya. Prostranstvennye zadachi po neustanovivshemusya dvizheniyu szhimaemoi zhidkosti (Three-dimensional Problems in Unsteady Motion of a Compressible Fluid).— Izdatel'stvo MGU. 1962.
18. Gubkin, K.E. Rasprostranenie razryvov v zvukovyk volnakh (Propagation of Discontinuities in Sound Waves).— PMM, Vol. 22, No. 4. 1958.
19. Stanyukovich, K.P.— DAN SSSR, Vol. 112, No. 4. 1957.
20. Zaslavskii, B.I. Ob otrazhenii sfericheskoi udarnoi volny v vode ot svobodnoi poverkhnosti (On the Reflection of a Spherical Shock Wave in Water from a Free Surface).— PMTF, No. 6. 1963.
21. Bulakh, B.M. Doctor Thesis, MGU. 1962.
22. Chernyi, G.G. Techeniya gaza s bol'shoi sverkhzvukovoi skorost'yu (Flow of Gas at Hypersonic Velocity).— Fizmatgiz. 1959.
23. Babich, V.M. Rasprostranenie nestatsionarnykh voln i kaustiki (Propagation of Nonstationary Waves and the Caustic).— Uchenye Zapiski LGU, No. 32. 1958.
24. Alekseev, A.S. O zadache tipa Lemba v lineino-neodnorodnom prostranstve (On Lamb's Problem in a Linear-inhomogeneous Space).— Uchenye Zapiski LGU, No. 32. 1958.
25. Yakovlev, Yu.S. Gidrodinamika vzryva (Hydrodynamics of Explosions).— Sudpromizdat. 1961.
26. Sears, W.R., ed. General Theory of High Speed Aerodynamics. N.J., Princeton Univ. Press. 1954;
Lighthill, M.J. Higher Approximations.
27. Bagdoev, A.G. Prostranstvennye nestatsionarnye dvizheniya sploshnoi sredy (Three-dimensional Nonstationary Motions of a Continuous Medium).— Erevan, Izdatel'stvo AN Arm SSR. 1961.
28. Moore, F.K. Second Approximation to Supersonic Conical Flows.— J. Aero. Sci., Vol. 17, p. 328. 1950.
29. Tan, H.S. Second Approximation to Conical Flows.— Ithaca, Cornell Univ. Graduate School of Aero. Engineering. 1950.
30. Bagdoev, A.G. Vzaimodeistvie vozmushchenii s udarnym frontom (Interaction of Disturbances with a Shock Front).— Izvestiya AN Arm SSR, Seriya Fiziko-matematicheskaya, Vol. 15, No. 6. 1962.

31. Bagdoev, A.G. O zakone dvizheniya udarnoi volny v zhidkosti (On the Law of Motion of a Shock Wave in a Fluid).— Izvestiya AN Arm SSR, Vol.17, No.3. 1964.
32. Bagdoev, A.G. Proniknovenie davleniya v neodnorodnuyu szhimaemuyu zhidkost' (Penetration of Pressure into an Inhomogeneous Compressible Fluid).— Izvestiya AN Arm SSR, Vol.16, No.3. 1963.
33. Bagdoev, A.G. Proniknovenie davleniya s perekhodom cherez skorost' zvuka (Penetration of Pressure with Transition through the Velocity of Sound).— Izvestiya AN Arm SSR, Vol.16, No.2. 1963.
34. Bagdoev, A.G. Opredelenie davleniya na udarnoi volne v zhidkosti (Determination of the Pressure on a Shock Wave in a Fluid).— Izvestiya AN Arm SSR, Seriya Fiziko-matematicheskaya, Vol.17, No.5. 1964.
35. Bagdoev, A.G. Ploskaya i osesimmetrichnaya zadacha dlya davleniya (Plane and Axisymmetric Problems for Pressure).— Izvestiya AN Arm SSR, Seriya Fiziko-matematicheskaya, Vol.14, No.5. 1961.
36. Bagdoev, A.G. Opredelenie zakona raspredeleniya davleniya, rasprostranyayushchegosya s dozvukovoi skorost'yu (Determination of the Distribution Law for Pressure Propagating at a Subsonic Velocity).— Izvestiya AN Arm SSR, Seriya Fiziko-matematicheskaya, Vol.16, No.4. 1962.
37. Bagdoev, A.G. Issledovanie raspredeleniya davleniya na udarnoi volne (Investigation of the Pressure Distribution on a Shock Wave).— Izvestiya AN Arm SSR, Seriya Fiziko-matematicheskaya, Vol.15, No.4. 1964.
38. Publ. Astron. Soc. Japan, Vol.14, No.3. 1962.
39. Lugovtsev, B.A. Rasprostranenie udarnoi volny (Shock Wave Propagation).— PMTF, No.3. 1952.
40. Phythian, J.E. The Energy Distribution behind a Decaying Two-dimensional Shock.— Quart. J. Mech. Appl. Math., Vol.5, p.318, 1952.
41. Clarkson, M.H. A Second-order Theory for Three-dimensional Wings in Supersonic Flow.— Quart. J. Mech. Appl. Math., Vol.7, p.203. 1954.
42. Grigoryan, S.S. On the Paper "Penetration of a Wedge into Compressible Half-space" by Z.N. Dobrovol'skaya.— PMM, Vol.26. 1962.
43. Whitham, G.B. On the Propagation of Weak Shock Waves.— J. Fluid Mech., Vol.1, p.290. 1956.
44. Lighthill, M.J. The Energy Distribution Behind Decaying Shocks I: Plane Waves.— Philos. Mag. Ser.7, Vol.41, p.1101. 1950.
45. Naugol'nykh, K., S.I. Soluyan, and R. Khokhlov.— Vestnik MGU, No.3, 52, 1961; Vestnik MGU, Seriya Fizicheskaya, No.4, 65. 1962.
46. Zel'dovich, Ya.B. Skhodyashchayasya udarnaya volna (Converging Shock Waves).— ZhETF, Vol.36, No.31, 1959.
47. Kuo, Y.H. A Similarity Rule for the Interaction between a Conical Field and a Plane Shock.— J. Aero. Sci., Vol.22, p.504. 1955.

48. Zvolinskii, N.V. and G.A.Skuridin. Ob asimptoticheskom metode resheniya dinamicheskikh zadach teorii uprugosti (On the Asymptotic Method of Solving Dynamic Problems of the Theory of Elasticity).— Izvestiya AN SSSR, Seriya Geofizicheskaya, No. 2. 1956.
49. Lighthill, M.J. Studies on Magnetohydrodynamic Waves.— Phil. Trans. Roy. Soc., Vol. 252. 1960.
50. Alekseev, A.S. and B.Ya.Bel'chinskii. Ob opredelenii intensivnosti voln i teorii uprugnosti luchevym metodom (Wave Intensity and the Theory of Elasticity Using a Ray Representation).— DAN SSSR, Vol. 118, No. 5.
51. Petrashen', G.I. et al.— Uchenye Zapiski LGU, No. 1. 1950.
52. Miles, J.W. On the Response of an Elastic Half-Space to a Moving Blast Wave.— J. Appl. Mech., Vol. 27, p. 710. 1960.
53. Yavorskii, B.M. and A.A.Detlaf. Spravochnik po fizike (Handbook of Physics), Moskva. 1963.
54. Pai, Shi-I. Magnetogasdynamics and Plasma Dynamics.— Prentice-Hall. 1962.
55. Courant, R. Partial Differential Equations.— New York— London, Interscience. 1962.
56. Lamb, H. Hydrodynamics.— Cambridge Univ. Press. 1959.
57. Whitham, G.B. Some Comments on Wave Propagation and Shock Wave Structure with Application to Magnetohydrodynamics.— Comm. Pure Appl. Math., Vol. 12. p. 113. 1959.
58. Babaev, D.A. Numerical Solution of a Flow Problem on the Upper Surface of a Supersonic Delta Wing.— Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki, Vol. 2, No. 6. 1962.
59. Fowell, L.R. Exact and Approximate Solutions for the Supersonic Delta Wing.— J. Aero. Sci., Vol. 23, p. 709. 1956.
60. Grib, A.A. Vestnik Leningradskogo Universiteta, Vol. 9, No. 4. 1964.
61. Ting, Lu. The Shock Strength in a Two-dimensional Nonsteady Flow.— J. Aeronaut. Sci., Vol. 19, p. 351. 1952.
62. Whitham, G.B. Non-linear Dispersive Waves.— Proc. Roy. Soc., Ser. A, Vol. 283, p. 238. 1965.
63. Polyanskii, O.Yu.— PMM, Vol. 24, No. 5. 1960.
64. Bagdoev, A.G. Reshenie zadachi o davlenii, prilozhennom k granitse zhidkogo poluprostranstva (Solution of the Problem of a Pressure Applied to the Boundary of a Fluid Half-space).— Izvestiya AN Arm SSR, Seriya Fiziko-matematicheskaya, No. 3. 1965.
65. Sugo. Transactions of Aero and Space Sciences of Japan, Vol. 2, No. 2. 1959.